HIGGS BUNDLES OVER A NONORIENTABLE SURFACE

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ABSTRACT. In this paper we define the notion of a Higgs bundle over a non-orientable manifold Σ whose orientable double cover $\tilde{\Sigma}$ is complex and use Higgs bundles to study the moduli space of representations of the fundamental group of Σ into a complex semisimple Lie group. When the orientable double cover $\tilde{\Sigma}$ is also compact and Kähler we prove that the Donaldson-Corlette correspondence holds on Σ. When $\tilde{\Sigma}$ is a compact Riemann surface we use this to (a) describe the moduli space of Higgs bundles on Σ as the fixed point set of an involution and (b) study the symplectic structure on the moduli space.

1. Introduction

Let M be a compact Kähler manifold and G a complex semisimple Lie group. Given a topologically trivial bundle $P \to M$ with structure group G, the nonabelian Hodge theory due to Hitchin, Simpson, Donaldson and Corlette ([Hi1], [S1], [S2], [D2], [C]) describes three homeomorphic moduli spaces:

- (Dolbeault) the moduli space of semistable Higgs bundles on P,
- (de Rham) the moduli space of flat connections on P, and
- (Betti) the moduli space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/\!\!/G$.

The homeomorphism between the Betti and de Rham moduli space is given by the well known correspondence between flat connections and their holonomy representations. To complete the picture we need another moduli space: the *Hitchin moduli space*, originally defined in [Hi1] via a hyperkähler quotient construction. The nonabelian Hodge theory then divides into two parts: Hitchin and Simpson identify the Dolbeault and Hitchin moduli spaces and Donaldson and Corlette identify the de Rham and Hitchin moduli spaces.

In this paper we prove an analog of the Donaldson-Corlette theorem for a nonorientable manifold whose orientable double cover is compact and Kähler. As a consequence we show that the smooth part of the associated Hitchin moduli space is the fixed point set of an involution on the Hitchin space for the orientable double cover, and that there is a symplectic structure on the subset of smooth points in this space.

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In order to state the main results we first need to outline the definition of a Higgs bundle on a nonorientable manifold (see Section 2 for more details). Let Σ be a nonorientable manifold, let $\tilde{\Sigma}$ be the orientable double cover of Σ with covering map $\pi : \tilde{\Sigma} \to \Sigma$ and suppose that $\tilde{\Sigma}$ has a complex structure. Let $P^{\mathbb{C}} \to \Sigma$ be a principal bundle with complex reductive structure group G, and let $\tilde{P}^{\mathbb{C}} := \pi^* P^{\mathbb{C}}$ denote the associated principal G-bundle over $\tilde{\Sigma}$. Pullback by π gives an injective map from connections on $P^{\mathbb{C}}$ to connections on $\tilde{P}^{\mathbb{C}}$. Let K be the maximal compact subgroup of G. A reduction of structure group from $P^{\mathbb{C}}$ to a principal K-bundle P induces a reduction of structure group from $\tilde{P}^{\mathbb{C}}$ to $\tilde{P} := \pi^* P$ (analogous to the extra data of a Hermitian metric of the fibres of a vector bundle). Given this reduction of structure group, there is a well-defined notion of a Higgs bundle on $\tilde{P}^{\mathbb{C}}$ (see Section 2.4 for more details) and our goal is to use this to define Higgs bundles on $P^{\mathbb{C}}$.

The nontrivial deck transformation $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ induces an involution on the space of connections on $\tilde{P}^{\mathbb{C}}$ which fixes the connections in the image of π^* (i.e. those coming from the nonorientable manifold). We consider the fixed points of this involution and define a Higgs bundle on $P^{\mathbb{C}} \to \Sigma$ to be a connection that pulls back to a Higgs bundle on $\tilde{P}^{\mathbb{C}}$. Equivalently, we can also write down the equations that a connection on $P^{\mathbb{C}}$ must satisfy to be a Higgs bundle (see (2.11)). Note that the definition of a Higgs bundle depends on the choice of reduction of structure group from $P^{\mathbb{C}}$ to P.

From the definition of a Higgs bundle, we can define the *space of Higgs bundles*, denoted $\mathcal{A}^{Higgs}$, and the space of solutions to Hitchin's equations (2.12), denoted $\mathcal{A}^{Hitchin}$. The *moduli space of solutions to Hitchin's equations* is then the quotient $\mathcal{M}^{Hitchin} := \mathcal{A}^{Hitchin}/\mathcal{G}$.

Recall from [G2] the definition of a reductive representation $\pi_1(\Sigma) \to G$. A flat connection on $P^{\mathbb{C}}$ is defined to be reductive if and only if the associated holonomy representation $\pi_1(\Sigma) \to G$ is reductive. The *Betti moduli space* is the quotient of the space of reductive representations by the conjugation action of G.

$$\mathcal{M}^{Betti}(\Sigma, G) := \mathrm{Hom}^{red}(\pi_1(\Sigma), G)/G$$

Inside the space $\mathcal{A}^{flat}$ of flat connections on $P^{\mathbb{C}}$ there is the space $\mathcal{A}^{flat,red}$ of flat, reductive connections. Taking the holonomy representation of a flat connection identifies $\mathcal{M}^{Betti}$ with the *de Rham moduli space*

$$\mathcal{M}^{dR} := \mathcal{A}^{flat,red}/\mathcal{G}^{\mathbb{C}}.$$

In Section 3 we prove the first main theorem, which is an analog of the Donaldson-Corlette correspondence for nonorientable manifolds. This allows us to describe the character variety of Σ via the theory of Higgs bundles.

Theorem 1.1. *Suppose that the orientable double cover of Σ is compact and Kähler. A flat connection D on $P^{\mathbb{C}} \to \Sigma$ is reductive if and only if there exists a reduction of structure group from $P^{\mathbb{C}}$ to P such that $D = d_A - i\psi$ is a Higgs bundle. If such a reduction of structure group exists then it is unique.*

Equivalently, given a fixed reduction of structure group from $P^{\mathbb{C}}$ to P, a flat connection D on $P^{\mathbb{C}} \to \Sigma$ is reductive if and only if there exists a complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$ such that $g \cdot D$ is a Higgs bundle. The gauge transformation g is unique up to the action of $\mathcal{G}$.

When $\tilde{\Sigma}$ is a compact Riemann surface, then a further investigation of the gradient flow using the methods of [DWW] shows that there is a $\mathcal{G}$-equivariant deformation retract from $\mathcal{A}^{flat,red}$ to $\mathcal{A}^{Hitchin}$, i.e. the induced map $\mathcal{M}^{Betti} \cong \mathcal{M}^{dR} \to \mathcal{M}^{Hitchin}$ is continuous.

Theorem 1.2. *There is a homeomorphism $\mathcal{M}^{Hitchin} \cong \mathcal{M}^{Betti}$.*

There is a natural involution on the character variety of the double cover $\tilde{\Sigma}$, given by the induced action of τ on $\pi_1(\tilde{\Sigma})$. In Section 4 we prove the next main theorem, which describes the character variety of Σ as a subset of the fixed point set of this involution.

Theorem 1.3. *Let G be a complex semi-simple Lie group. The fixed point set $(\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\mathrm{smooth}}/G)^\tau$ of the smooth moduli space $\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\mathrm{smooth}}/G$ under the involution τ consists of complex submanifolds $\mathcal{N}_0$, $\mathcal{N}_{\lambda_r}$ for $1 \leq \lambda_r \leq |Z(G)/2Z(G)| - 1$, where*

1. *$\mathcal{N}_0 \stackrel{\mathrm{def}}{=} P_r(N_{e_G}^{\mathrm{smooth}})$, $\mathcal{N}_{\lambda_r} \stackrel{\mathrm{def}}{=} P_r(N_r^{\mathrm{smooth}})$, and λ_r is the image of $r \in Z(G)$ under the map $Z(G)/2Z(G) \to |Z(G)/2Z(G)| \subset \mathbb{N} \cup \{0\}$ where the identity element e_G is sent to 0.*
2. *The smooth moduli space $\mathrm{Hom}(\pi_1(\Sigma), G)^{\mathrm{simp}} /\!\!/ G$ over the nonorientable surface is a $|Z(G)/2Z(G)| : 1$ covering space of the submanifold $\mathcal{N}_0$. In particular, if $|Z(G)|$ is odd, then $\mathrm{Hom}(\pi_1(\Sigma), G)^{\mathrm{simp}} /\!\!/ G$ is homeomorphic to the fixed point set $(\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\mathrm{smooth}}/G)^\tau$ since it only contains $\mathcal{N}_0$ in this case.*
3. *Assume further that G is simply connected. Then the Betti moduli space $\mathrm{Hom}(\pi_1(\Sigma), PG) /\!\!/ PG$ for the nonorientable surface Σ with structure group the projective group $PG = G/Z(G)$ is a disjoint union of $\rho(C_r /\!\!/ G)$ for all $r \in Z(G)/2Z(G)$, where ρ is the universal covering map $G \to PG$. Moreover, each $C_r^{\mathrm{simp}} /\!\!/ G$ is a $|Z(G)/2Z(G)| : 1$ covering space of the submanifold $\mathcal{N}_{\lambda_r}$.*

Theorem 1.1 above shows that the character variety of Σ corresponds to a symplectic quotient of the space of flat connections. This leads to the next theorem,

proved in Section 5, which shows that the moduli space of solutions to Hitchin's equations has a symplectic structure.

Theorem 1.4. *The smooth part of $\mathcal{M}^{Hitchin}$ is a symplectic manifold.*

In view of the theorem above, it is natural to ask whether there are any smooth points in $\mathcal{M}^{Hitchin} \cong \mathcal{M}^{dR} \cong \mathcal{M}^{Betti}$ and, if so, how to characterise them. In other words, we want to study the deformation theory of the character variety of Σ. This is the main result of the paper [HW], where we give criteria for points in $\mathcal{M}^{Betti} \cong \mathcal{M}^{dR}$ to be smooth.

There are also a number of other interesting questions that remain unexplored. Firstly, in analogy with the nonabelian Hodge theory for compact orientable Kähler manifolds, one would like to study the quotient of $\mathcal{A}^{Higgs}$ by the action of the complex gauge group, where the action is the usual one on the space of Higgs bundles from [Hi1] (note that this is different to the natural action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}^{flat}$ from [C] that is used in this paper). One can also define Higgs bundles with respect to real reductive groups as in [Hi2] and study these moduli spaces.

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2. Preliminary and setup

2.1. **An involution on the space of flat connections.** Let Σ be a nonorientable manifold and let $\pi : \tilde{\Sigma} \to \Sigma$ denote its orientable double cover such that $\tilde{\Sigma}/\tau \cong \Sigma$ where τ is the nontrivial deck transformation on $\tilde{\Sigma}$. For the definition of a Higgs bundle we assume that $\tilde{\Sigma}$ is compact and complex, for the Donaldson-Corlette theorem in Section 3 we assume that $\tilde{\Sigma}$ is compact and Kähler, and for the results of Sections 4 and 5 we assume that $\tilde{\Sigma}$ is a compact Riemann surface.

Let K be a compact connected semi-simple Lie group and let G denote its complexification. Let P be a principal K-bundle over Σ and $\tilde{P} := \pi^* P$ the pull back bundle over $\tilde{\Sigma}$. Denote by $\mathcal{A} =: \mathcal{A}(P) \cong \Omega^1(\Sigma, \mathrm{ad} P)$ the space of all connection 1-forms on P and $\tilde{\mathcal{A}} =: \mathcal{A}(\tilde{P}) \cong \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$ the space of all connection 1-forms on $\tilde{P}$. (Note that *a priori* $\mathcal{A}(P)$ and $\mathcal{A}(\tilde{P})$ are affine spaces, and the identifications given above involve the choice of a connection.) We use $P^{\mathbb{C}}$ and $\tilde{P}^{\mathbb{C}}$ to denote the associated bundles with structure group $G = K^{\mathbb{C}}$.

The cotangent bundle of $\mathcal{A}(P)$ is

$$T^*\mathcal{A} =: T^*\mathcal{A}(P) \cong \{(A, \alpha) \in T^*\mathcal{A}(P) \mid A, \alpha \in \Omega^1(\Sigma, \mathrm{ad} P)\}$$

and the cotangent bundle of $\mathcal{A}(\tilde{P})$ is

$$T^*\tilde{\mathcal{A}} =: T^*\mathcal{A}(\tilde{P}) \cong \{(\theta,\xi) \in T^*\mathcal{A}(\tilde{P}) \mid \theta, \xi \in \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P})\}.$$

Moreover, one may say that $A + \sqrt{-1}\alpha \in \Omega^1(\Sigma, \mathrm{ad}P^{\mathbb{C}}) \cong \mathcal{A}(P^{\mathbb{C}})$ and $\theta + \sqrt{-1}\xi \in \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P}^{\mathbb{C}}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ where again we have chosen a connection for these identifications. Thus we have identifications $T^*\mathcal{A}(P) \cong \Omega^1(\Sigma, \mathrm{ad}P^{\mathbb{C}})$ and $T^*\mathcal{A}(\tilde{P}) \cong \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P}^{\mathbb{C}})$.

The nontrivial deck transformation τ on $\tilde{\Sigma}$ induces many involutions on $\tilde{P}$. Choose $\tau : \tilde{P} \to \tilde{P}$ such that $\tilde{P}/\tau \cong P$. This involution τ induces involution τ^* on $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$.

The projection $\pi : \tilde{\Sigma} \to \Sigma$ induces an inclusion $\pi^* : \Omega^*(\Sigma, \mathrm{ad}P) \hookrightarrow \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$ and it interacts with the induced involution τ^* on $\mathcal{A}(\tilde{P})$, $T^*\mathcal{A}(\tilde{P})$, and $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$ as follows:

$$\tau^* : \mathcal{A}(\tilde{P}) \to \mathcal{A}(\tilde{P}), \quad (\mathcal{A}(\tilde{P}))^{\tau^*} = \pi^*(\mathcal{A}(P)) \cong \mathcal{A}(P)$$
$$\tau^* : T^*\mathcal{A}(\tilde{P}) \to T^*\mathcal{A}(\tilde{P}), \quad (T^*\mathcal{A}(\tilde{P}))^{\tau^*} = \pi^*(T^*\mathcal{A}(P)) \cong T^*\mathcal{A}(P)$$
$$\tau^* : \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P}) \to \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P}), \quad (\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P}))^{\tau^*} = \pi^*(\Omega^*(\Sigma, \mathrm{ad}P)) \cong \Omega^*(\Sigma, \mathrm{ad}P)$$

Lemma 2.1. *Let (M,g) be a compact orientable Riemannian manifold of dimension m. Let τ be an orientation reversing isometric involution on M and $\star$ the Hodge star operator on M. Then $\tau^* \star \theta = - \star \tau^*\theta$ for all $\theta \in \Omega^*(M)$.*

Proof. we know that $\star : \Omega^k(M) \to \Omega^{m-k}(M)$, $\theta \mapsto \star\, \theta$ is defined by $\theta \wedge \star\, \theta = g(\theta,\theta)\,\mathrm{dvol}(g)$. Since τ is an orientation reversing isometric involution on M, we have

$$(\tau^*\theta) \wedge (\tau^* \star\ \theta) = \tau^*(\theta \wedge \star\, \theta) = -\theta \wedge \star\, \theta = -g(\theta,\theta)\,\mathrm{dvol}(g).$$

On the other hand, we know that

$$(\tau^*\theta) \wedge \star\, (\tau^*\theta) = g(\tau^*\theta, \tau^*\theta)\,\mathrm{dvol}(g) = g(\theta,\theta)\,\mathrm{dvol}(g)$$

by the definition of the $\star$ operator and τ being isometry. Thus, we have

$$(\tau^*\theta) \wedge (\tau^* \star\ \theta) = -(\tau^*\theta) \wedge (\star\, \tau^*\theta), \quad \text{i.e.} \quad \tau^* \star\ \theta = - \star\ \tau^*\theta$$

for all $\theta \in \Omega^*(M)$. □

A similar argument shows that this is also true for any $\theta \in \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$.

Lemma 2.2. *Given $\theta \in \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$ and $\xi \in \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$, we have*

1. $\tau^*(d_\theta \xi) = d_{\tau^*\theta}\tau^*\xi$.
2. $\tau^*(\star\, d_\theta \star \xi) = \star\, d_{\tau^*\theta} \star \tau^*\xi$.
3. $\tau^*(d_\theta \star \xi) = -d_{\tau^*\theta} \star \tau^*\xi$.

where $\star$ is the Hodge star operator on $\tilde{\Sigma}$.

Proof. (1) $\tau^*(d_\theta \xi) = \tau^* d\xi + \tau^*[\theta, \xi] = d\tau^*\xi + [\tau^*\theta, \tau^*\xi] = d_{\tau^*\theta}\tau^*\xi$.

(2)

$$\begin{aligned}\tau^*(\star\, d_\theta \star \xi) &= \tau^*(\star\, d \star \xi + \star[\theta, \star\xi]) \\ &= -\star\, \tau^*(d\star \xi) - \star\,\tau^*[\theta, \star\xi] \qquad \text{because } \tau^*\star = -\star\, \tau^* \\ &= \star\, d \star \tau^*\xi + \star[\tau^*\theta, \star\,\tau^*\xi] \qquad \text{used again } \tau^*\star = -\star\, \tau^* \\ &= \star\, d_{\tau^*\theta}(\star\,\tau^*\xi)\end{aligned}$$

(3)

$$\begin{aligned}\tau^*(d_\theta \star \xi) &= \tau^*(d \star \xi + [\theta, \star\xi]) \\ &= d\tau^*(\star\xi) + \tau^*[\theta, \star\xi] \\ &= -d \star \tau^*\xi - [\tau^*\theta, \star\,\tau^*\xi] \qquad \text{used } \tau^*\star = -\star\, \tau^* \\ &= -d_{\tau^*\theta}(\star\,\tau^*\xi)\end{aligned}$$

□

Given $A \in \Omega^1(\Sigma, \mathrm{ad}P)$ and $\alpha \in \Omega^*(\Sigma, \mathrm{ad}P)$, let $\tilde{A}$ and $\tilde{\alpha}$ denote the respective lifts to $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$, i.e. $\pi^*(A) = \tilde{A}$ and $\pi^*(\alpha) = \tilde{\alpha}$. Lemma 2.2 implies that $\tau^*(d_{\tilde{A}}\tilde{\alpha}) = d_{\tilde{A}}\tilde{\alpha}$ and $\tau^*(d^*_{\tilde{A}}\tilde{\alpha}) = d^*_{\tilde{A}}\tilde{\alpha}$, since $\tilde{A}$ and $\tilde{\alpha}$ are both τ^*-invariant. Moreover, $\widetilde{d_A\alpha} = d_{\tilde{A}}\tilde{\alpha}$ if we define $d_A\alpha =: d\alpha + [A, \alpha]$:

$$\begin{aligned}\pi^*(d_A\alpha) &= \pi^*(d\alpha + [A, \alpha]) = d\pi^*\alpha + \pi^*[A, \alpha] = d(\pi^*\alpha) + [\pi^*A, \pi^*\alpha] \\ &= d_{\pi^*A}(\pi^*\alpha) = d_{\tilde{A}}\tilde{\alpha}\end{aligned}$$

From the above discussion, we can make the following definition:

Definition 2.3. *Given $A \in \Omega^1(\Sigma, \mathrm{ad}P)$, the operators d_A, $d_A^* : \Omega^*(\Sigma, \mathrm{ad}P) \to \Omega^*(\Sigma, \mathrm{ad}P)$ are defined by*

$$\begin{aligned}d_A\alpha &:= d\alpha + [A, \alpha] \\ d_A^*\alpha &:= (\pi^*)^{-1}(d^*_{\tilde{A}}\tilde{\alpha})\end{aligned}$$

for any $\alpha \in \Omega^*(\Sigma, \mathrm{ad}P)$. (Recall that $\pi^* : \Omega^*(\Sigma, \mathrm{ad}P) \to \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$ is injective, so the inverse $(\pi^*)^{-1}$ is defined on the image of π^*.)

On the space $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$, there is a well-defined inner product $< \cdot, \cdot >_{\tilde{\Sigma}}$ (cf: [AB], [Hi1]):

$$< \xi, \eta >_{\tilde{\Sigma}} \stackrel{\text{def}}{=} \int_{\tilde{\Sigma}} \xi \wedge \bar{\star}\, \eta \stackrel{\text{def}}{=} \int_{\tilde{\Sigma}} Tr(\xi \wedge \star\, \eta^*), \qquad \xi, \eta \in \Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$$

where η^* means that one takes the complex conjugate of the Lie algebra part. For example, $K = U(n)$ or $SU(n)$, $< \xi, \eta >_{\tilde{\Sigma}} = -\int_{\tilde{\Sigma}} \mathrm{Tr}(\xi \wedge \star\, \eta)$ since their Lie algebras are skew-Hermitian matrices.

If we take the L^2 completion $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})_{L^2}$, then the inner product $< \xi, \eta >_{\tilde{\Sigma}}$ gives this the structure of a Hilbert space, and the τ^*-invariant part $\Omega^*(\tilde{\Sigma}, \mathrm{ad}\tilde{P})^{\tau^*}$

is also a Hilbert space since $\Omega^*(\tilde\Sigma, \mathrm{ad}\tilde P)^{\tau^*}$ is a closed linear subspace of $\Omega^*(\tilde\Sigma, \mathrm{ad}\tilde P)$. Thus, it makes sense to define an inner product on $\Omega^*(\Sigma, \mathrm{ad}P)$ via the inner product on $\Omega^*(\tilde\Sigma, \mathrm{ad}\tilde P)$ since one can identify $\Omega^*(\Sigma, \mathrm{ad}P)$ with the τ^* invariant forms $\Omega^*(\tilde\Sigma, \mathrm{ad}\tilde P)^{\tau^*}$.

Definition 2.4. *The inner product on $\Omega^*(\Sigma, \mathrm{ad}P)$ is defined by*

$$< \alpha, \beta >_\Sigma =: \int_{\tilde\Sigma} \tilde\alpha \wedge \bar\star\, \tilde\beta \qquad \alpha, \beta \in \Omega^*(\Sigma, \mathrm{ad}P)$$

Lemma 2.5. *The operator d_A^* defined in Definition 2.3 is the adjoint of d_A with respect to this inner product. In other words, $< d_A\alpha, \beta >_\Sigma = < \alpha, d_A^*\beta >_\Sigma$.*

Proof.

$$\begin{aligned} < d_A\alpha, \beta >_\Sigma &= \int_{\tilde\Sigma} \widetilde{d_A\alpha} \wedge \bar\star\, \tilde\beta \\ &= \int_{\tilde\Sigma} d_{\tilde A}\tilde\alpha \wedge \bar\star\, \tilde\beta && \text{by definition of } d_A \\ &= \int_{\tilde\Sigma} \tilde\alpha \wedge \bar\star\, d_{\tilde A}^*\tilde\beta && d_A^* \text{ is the adjoint of } d_A \text{ on } \Omega^*(\tilde\Sigma, \mathrm{ad}\tilde P) \\ &= \int_{\tilde\Sigma} \tilde\alpha \wedge \bar\star\, \widetilde{d_A^*\beta} && \text{by definition of } d_A^* \\ &= < \alpha, d_A^*\beta >_\Sigma \end{aligned}$$

Thus, d_A is the adjoint of d_A^* with respect to the inner product $< \cdot, \cdot >_\Sigma$. ∎

Proposition 2.6. *The energy functional*

$$f : T^*\tilde{\mathcal{A}} \to \mathbb{R}, \quad (\theta, \xi) \mapsto \|d_\theta^*\xi\|^2 = < d_\theta^*\xi, d_\theta^*\xi >_{\tilde\Sigma}$$

is τ^*-invariant, for any $(\theta, \xi) \in T^*\tilde{\mathcal{A}}$; i.e., $f(\tau^*\theta, \tau^*\xi) = f(\theta, \xi)$, $\forall(\theta, \xi) \in T^*\tilde{\mathcal{A}}$.

Proof.

$$\begin{aligned} f(\tau^*\theta, \tau^*\xi) &= \int_{\tilde\Sigma} d_{\tau^*\theta}^*\tau^*\xi \wedge \bar\star\, d_{\tau^*\theta}^*\tau^*\xi \\ &= \int_{\tilde\Sigma} \tau^*(d_\theta^*\xi) \wedge \bar\star\, \tau^*(d_\theta^*\xi) && \text{by Lemma 2.2} \\ &= -\int_{\tilde\Sigma} \tau^*(d_\theta^*\xi \wedge \bar\star\, d_\theta^*\xi) && \text{since } \tau^*\star = -\star\tau^* \\ &= \int_{\tilde\Sigma} d_\theta^*\xi \wedge \bar\star\, d_\theta^*\xi && \text{since } \tau \text{ is an orientation reversing isometry on } \tilde\Sigma \\ &= < d_\theta^*\xi, d_\theta^*\xi >_{\tilde\Sigma} = f(\theta, \xi). \end{aligned}$$

∎

In fact, more is true: If a function F is defined by $F(\omega) =: \int_{\tilde\Sigma} \omega \wedge \bar\star\, \omega$ whenever this integral makes sense and τ is an orientation reversing isometry on $\tilde\Sigma$, then

$$F(\tau^*\omega) = \int_{\tilde\Sigma} \tau^*\omega \wedge \bar\star\, \tau^*\omega = -\int_{\tilde\Sigma} \tau^*(\omega \wedge \bar\star\, \omega) = \int_{\tau(\tilde\Sigma)} \omega \wedge \bar\star\, \omega = \int_{\tilde\Sigma} \omega \wedge \bar\star\, \omega = F(\omega).$$

In other words, F is always τ^*-invariant.

Let $\mathcal{G}$ (respectively $\tilde{\mathcal{G}}$, $\mathcal{G}^{\mathbb{C}}$ and $\tilde{\mathcal{G}}^{\mathbb{C}}$) denote the gauge group of P (respectively $\tilde{P}$, $P^{\mathbb{C}}$ and $\tilde{P}^{\mathbb{C}}$). We can identify the gauge groups $\mathcal{G}$ of P and gauge groups $\mathcal{G}^{\mathbb{C}}$ of $P^{\mathbb{C}}$ with the subgroups of τ-invariant gauge transformations in $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}^{\mathbb{C}}$ respectively, i.e. $\mathcal{G} \cong \tilde{\mathcal{G}}_\tau$ and $\mathcal{G}^{\mathbb{C}} \cong \tilde{\mathcal{G}}^{\mathbb{C}}_\tau$.

2.2. **Representations and the involution.** Let Σ_0^ℓ be the closed, compact, connected, orientable surface with $\ell \geq 0$ handles. Let Σ_1^ℓ be the connected sum of Σ_0^ℓ and $\mathbb{RP}^2$, and let Σ_2^ℓ be the connected sum of Σ_0^ℓ and a Klein bottle. Any closed, compact, connected surface is of the form Σ_i^ℓ, where ℓ is a nonnegative integer, $i = 0, 1, 2$. Σ_i^ℓ is orientable if and only if $i = 0$. Use 1 as the identity of $\pi_1(\Sigma)$. We have

$$
\begin{aligned}
\pi_1(\Sigma_0^\ell) &= \langle A_1, B_1, \ldots, A_\ell, B_\ell \mid \prod_{i=1}^{\ell} [A_i, B_i] = 1 \rangle \\
\pi_1(\Sigma_1^\ell) &= \langle A_1, B_1, \ldots, A_\ell, B_\ell, C \mid \prod_{i=1}^{\ell} [A_i, B_i] = C^2 \rangle \\
\pi_1(\Sigma_2^\ell) &= \langle A_1, B_1, \ldots, A_\ell, B_\ell, D, C \mid \prod_{i=1}^{\ell} [A_i, B_i] = CDC^{-1}D \rangle
\end{aligned}
$$

Let $\Sigma = \Sigma_i^\ell$, where $i = 1, 2$. Then Σ is homeomorphic to the connected sum of $2\ell + i$ copies of $\mathbb{RP}^2$, and its orientable double cover $\tilde{\Sigma}$ is $\Sigma_0^{2\ell+i-1}$, a Riemann surface of genus $2\ell + i - 1$.

Notation 2.7. *In the rest of this paper, we will use the following notation:*

Denote the 2ℓ-vector by $V = (a_1, b_1, \ldots, a_\ell, b_\ell) \in G^{2\ell}$. Define $\mathfrak{m}(V)$ and $\mathfrak{r}(V)$ by

$$
\mathfrak{m}(V) = \prod_{i=1}^{\ell} [a_i, b_i] \tag{2.1}
$$

$$
\mathfrak{r}(V) = (b_\ell, a_\ell, \ldots, b_1, a_1) \tag{2.2}
$$

Then $\mathfrak{m}(\mathfrak{r}(V)) = \mathfrak{m}(V)^{-1}$. Let gVg^{-1} denote $(ga_1g^{-1}, gb_1g^{-1}, \ldots, ga_\ell g^{-1}, gb_\ell g^{-1})$ for $g \in G$.

With the above notation, the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), G)$ can be written as follows:

$$
\begin{aligned}
\mathrm{Hom}(\pi_1(\Sigma_0^\ell), G) &= \{V \in G^{2\ell} \mid \mathfrak{m}(V) = e_G\} \\
\mathrm{Hom}(\pi_1(\Sigma_1^\ell), G) &= \{(V, c) \mid V \in G^{2\ell},\ c \in G,\ \mathfrak{m}(V) = c^2\} \\
\mathrm{Hom}(\pi_1(\Sigma_2^\ell), G) &= \{(V, d, c) \mid V \in G^{2\ell},\ d, c \in G,\ \mathfrak{m}(V) = cdc^{-1}d\}
\end{aligned}
$$

where e_G denotes the identity element of G.

The symmetric representation varieties of flat connections are

$$\begin{aligned}
Z^{\ell,1}_{\text{flat}}(G) &= \{(V,c,V',c') \mid V,V' \in G^{2\ell},\ c,c' \in G,\ \mathfrak{m}(V) = cc', \mathfrak{m}(V') = c'c\} \\
Z^{\ell,2}_{\text{flat}}(G) &= \{(V,d,c,V',d',c') \mid V,V' \in G^{2\ell},\ d,c,d',c' \in G,\ \mathfrak{m}(V) = cd'c^{-1}d, \\
&\qquad \mathfrak{m}(V') = c'dc'^{-1}d'\}
\end{aligned}$$

We call this the *two base point Hom space.* Let $G \times G$ act on $Z^{\ell,1}_{\text{flat}}(G)$, $Z^{\ell,2}_{\text{flat}}(G)$ by

$$(g_1,g_2)\cdot(V,c,V',c') = (g_1Vg_1^{-1}, g_1cg_2^{-1}, g_2V'g_2^{-1}, g_2c'g_1^{-1}),$$
$$(g_1,g_2)\cdot(V,d,c,V',d',c') = (g_1Vg_1^{-1}, g_1dg_1^{-1}, g_1cg_2^{-1}, g_2V'g_2^{-1}, g_2d'g_2^{-1}, g_2c'g_1^{-1}),$$

respectively, where $V,V' \in G^{2\ell}$ and $g_1,g_2,c,c',d,d' \in G$.

Recall Lemma 2.3 and 2.4 from [HL]:

Lemma 2.8. *For $i = 1,2$, define $\Phi^{\ell,i}_G : G^{2(2\ell+i)} \to G^{2(2\ell+i-1)}$ by*

$$\begin{aligned}
\Phi^{\ell,1}_G(V,c,V',c') &= (V, c\,\mathfrak{r}(V')\,c^{-1}), \\
\Phi^{\ell,2}_G(V,d,c,V',d',c') &= (V, d^{-1}c\,\mathfrak{r}(V')\,c^{-1}d, d^{-1}, cc').
\end{aligned}$$

where $V,V' \in G^{2\ell}$, $c,d,c',d' \in G$. Then

$$\Phi^{\ell,i}_G(Z^{\ell,i}_{\text{flat}}(G)) = \text{Hom}(\pi_1(\Sigma_0^{2\ell+i-1}), G).$$

Since G is not necessarily a compact Lie group, not all of the orbits in $\text{Hom}(\pi_1(\Sigma), G)$ are closed. In order for the quotient spaces to be Hausdorff, we need to take the categorical quotients $\text{Hom}(\pi_1, G)/\!\!/G$ and $Z^{\ell,i}_{\text{flat}}(G)/\!\!/G \times G$, where the quotient $/\!\!/$ identifies orbits whose closures intersect.

A priori, a point in $\text{Hom}(\pi_1, G)$ that corresponds to a point in $Z^{\ell,i}_{\text{flat}}(G)$ whose $G \times G$-orbit is closed may itself have a non-closed orbit in G, and vice versa. However, the relation between the two spaces gives us the following lemmas:

Lemma 2.9. *The $G \times G$-orbit of $(V,c,V',c') \in Z^{\ell,1}_{\text{flat}}(G)$ is closed if and only if the G-orbit of $\Phi^{\ell,1}_G(V,c,V',c') = (V, c\,\mathfrak{r}(V')\,c^{-1})$ is closed.*

Proof. Suppose the $G \times G$-orbit of $(V,c,V',c') \in Z^{\ell,1}_{\text{flat}}(G)$ is closed. We want to show that, if $\{g_n\}$ is any sequence in G such that $\lim_{n\to\infty} g_n(V, c\,\mathfrak{r}(V')\,c^{-1})g_n^{-1}$ exists, then there exists an $h \in G$ such that

$$\lim_{n\to\infty} g_n(V, c\,\mathfrak{r}(V')\,c^{-1})g_n^{-1} = h((V, c\,\mathfrak{r}(V')\,c^{-1})h^{-1}$$

(since if the limit doesn't exist, then closedness is automatically satisfied).

Since all the entries in $\lim_{n\to\infty} g_n c\mathfrak{r}(V')c^{-1}g_n^{-1}$ exist, their product

$$\lim_{n\to\infty} g_n c\mathfrak{m}(V')c^{-1}g_n = \lim_{n\to\infty} g_n cc'g_n$$

exist also (where we used the relation that $\mathfrak{m}(V') = c'c$), and $\lim_{n\to\infty} g_n(cc')^{-1}g_n^{-1}$ exists as well. Thus

$$\lim_{n\to\infty} g_n(c')^{-1}V'c'g_n^{-1} = \lim_{n\to\infty} (g_n(c')^{-1}c^{-1}g_n^{-1})(g_ncV'c^{-1}g_n^{-1}))g_ncc'g_n^{-1})$$

exists also.

Define two sequences $\{g_n^1 = g_n\}$ and $\{g_n^2 = g_n(c')^{-1}\}$ in G. Then

$$\begin{aligned}
\lim_{n\to\infty}(g_n^1)V(g_n^1)^{-1} &= \lim_{n\to\infty}(g_n)V(g_n)^{-1}\\
\lim_{n\to\infty}(g_n^1)c(g_n^2)^{-1} &= \lim_{n\to\infty}(g_n)cc'(g_n)^{-1}\\
\lim_{n\to\infty}(g_n^2)V'(g_n^2)^{-1} &= \lim_{n\to\infty}(g_n)(c')^{-1}V'(c')(g_n)^{-1}\\
\lim_{n\to\infty}(g_n^2)c'(g_n^1)^{-1} &= \lim_{n\to\infty}(g_n)(c')^{-1}c'(g_n)^{-1} = e_G
\end{aligned}$$

all exist. Since the $G\times G$-orbit of (V,c,V',c') is closed, it means that there exists an $(h_1,h_2)\in G\times G$ such that

$$\lim_{n\to\infty}(g_n^1,g_n^2)\cdot(V,c,V',c') = (h_1,h_2)\cdot(V,c,V',c').$$

In other words, we have

$$\begin{aligned}
\lim_{n\to\infty}(g_n^1)V(g_n^1)^{-1} &= \lim_{n\to\infty}(g_n)V(g_n)^{-1} = h_1Vh_1^{-1}\\
\lim_{n\to\infty}(g_n^1)c(g_n^2)^{-1} &= \lim_{n\to\infty}(g_n)cc'(g_n)^{-1} = h_1ch_2^{-1}\\
\lim_{n\to\infty}(g_n^2)V'(g_n^2)^{-1} &= \lim_{n\to\infty}(g_n)(c')^{-1}V'(c')(g_n)^{-1} = h_2V'h_2^{-1}\\
\lim_{n\to\infty}(g_n^2)c'(g_n^1)^{-1} &= \lim_{n\to\infty}(g_n)(c')^{-1}c'(g_n)^{-1} = e = h_2c'h_1^{-1}
\end{aligned}$$

which gives

$$\begin{aligned}
\lim_{n\to\infty}g_ncV'c^{-1}g_n^{-1} &= \lim_{n\to\infty}(g_ncc'g_n^{-1})(g_n(c')^{-1}V'c'g_n^{-1})(g_n(cc')^{-1}g_n^{-1})\\
&= h_1ch_2^{-1}h_2V'h_2^{-1}h_2c^{-1}h_1^{-1} = h_1(cV'c^{-1})h_1^{-1}.
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty}g_n(V,cV'c^{-1})g_n^{-1} = h_1(V,cV'c^{-1})h_1^{-1},$$

and since $\{g_n\}$ is any sequence in G, this proves that the G-orbit of $(V,cV'c^{-1})$ is closed.

Now suppose that the G-orbit of $(V,cV'c^{-1})$ is closed. We want to show that the $G\times G$-orbit of (V,c,V',c') is closed. Let $\{g_n^1\}$ and $\{g_n^2\}$ be two sequences in G. Assume that $\lim_{n\to\infty}(g_n^1,g_n^2)\cdot(V,c,V',c')$ exists. Then

$$\lim_{n\to\infty}g_n^1\cdot(V,cV'c^{-1}) = \lim_{n\to\infty}(g_n^1V(g_n^1)^{-1},(g_n^1c(g_n^2)^{-1})(g_n^2V'(g_n^2)^{-1})(g_n^2c^{-1}(g_n^1)^{-1}))$$

exists also, and since the G-orbit of $(V,cV'c^{-1})$ is closed, this implies that there is an $h\in G$ such that $\lim_{n\to\infty}g_n^1\cdot(V,cV'c^{-1}) = h\cdot(V,cV'c^{-1})$. Clearly

$$\lim_{n\to\infty}\mathfrak{m}(g_n^1V(g_n^1)^{-1}) = \mathfrak{m}(hVh^{-1}) = h\mathfrak{m}(V)h^{-1}.$$

Since the limit exists then we may assume that $\lim_{n\to\infty} g_n^1 c(g_n^2)^{-1} = z \in G$. Let $k = z^{-1}hc \in G$. Then

$$\begin{aligned}
\lim_{n\to\infty} g_n^1 V(g_n^1)^{-1} &= hVh^{-1}\\
\lim_{n\to\infty} g_n^1 c(g_n^2)^{-1} &= z = hck^{-1}\\
\lim_{n\to\infty} g_n^2 V'(g_n^2)^{-1} &= \lim_{n\to\infty} g_n^2 c^{-1}(g_n^1)^{-1}(g_n^1 cV'c^{-1}(g_n^1)^{-1})g_n^1 c(g_n^2)^{-1}\\
&= Z^{-1}(hcV'c^{-1}h^{-1})Z\\
&= kc^{-1}h^{-1}(hcV'c^{-1}h^{-1})hck^{-1} = kV'k^{-1}\\
\lim_{n\to\infty} g_n^2 c'(g_n^1)^{-1} &= \lim_{n\to\infty} g_n^2 c^{-1}(g_n^1)^{-1}(g_n^1 cc'(g_n^1)^{-1})\\
&= \lim_{n\to\infty} g_n^2 c^{-1}(g_n^1)^{-1}(g_n^1 \mathfrak{m}(V)(g_n^1)^{-1})\\
&= \lim_{n\to\infty} g_n^2 c^{-1}(g_n^1)^{-1}\mathfrak{m}(g_n^1 V(g_n^1)^{-1}) = Z^{-1}\mathfrak{m}(hVh^{-1})\\
&= kc^{-1}h^{-1}hcc'h^{-1} = kc'h^{-1}
\end{aligned}$$

in other words,

$$\lim_{n\to\infty}(g_n^1, g_n^2)\cdot(V, c, V'c') = (h, k)\cdot(V, c, V', c').$$

Thus, the $G\times G$-orbit of $(V, c, V'c')$ is closed. $\square$

A similar argument shows that

Lemma 2.10. *The $G\times G$-orbit of $(V, d, c, V', d', c') \in Z^{\ell,2}_{\text{flat}}(G)$ is closed if and only if the G-orbit of $\Phi^{\ell,2}_G(V, d, c, V', d', c') = (V, d^{-1}c\,\mathfrak{r}(V')\,c^{-1}d, d^{-1}, cc')$ is also closed.*

Thus, combining Lemma 2.8, 2.9, and 2.10 implies

Proposition 2.11. *The map $\Phi^{\ell,i}_G : G^{2(2\ell+i)} \to G^{2(2\ell+i-1)}$ defined above induces homeomorphisms*

$$Z^{\ell,i}_{\text{flat}}(G)/\!\!/(G\times G) \cong \mathrm{Hom}(\pi_1(\Sigma_0^{2\ell+i-1}), G)/\!\!/G. \tag{2.3}$$

where the quotient $/\!\!/$ on both sides means that we identify orbits whose closures intersect to form the quotient space.

The involution τ acts on $Z^{\ell,i}_{\text{flat}}(G)$, $G\times G$ and $Z^{\ell,i}_{\text{flat}}(G)/\!\!/(G\times G)$ by

$$\begin{aligned}
&\tau(V, c, V', c') = (V', c', V, c);\quad \tau(V, d, c, V', d', c') = (V', d', c', V, d, c)\\
&\tau(g_1, g_2) = (g_2, g_1),\\
&\tau([(V, c, V', c')]) = [(V', c', V, c)];\quad \tau([(V, d, c, V', d', c')]) = [(V', d', c', V, d, c)]
\end{aligned}$$

Then $\mathrm{Hom}(\pi_1(\Sigma_i^\ell), G)/\!\!/G \cong Z^{\ell,i}_{\text{flat}}(G)^\tau/\!\!/(G\times G)^\tau$ and there is a natural map I:

$$\begin{aligned}
I:\ \mathrm{Hom}(\pi_1(\Sigma), G)/\!\!/G &\to (Z^{\ell,i}_{\text{flat}}(G)/\!\!/G\times G)^\tau \cong (\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)/\!\!/G)^\tau\\
[(V, c)] &\mapsto [(V, c, V, c)] \cong [(V, c\mathfrak{r}(V')c^{-1})]\\
[(V, d, c)] &\mapsto [(V, d, c, V, d, c)] \cong [(V, d^{-1}c\,\mathfrak{r}(V)\,c^{-1}d, d^{-1}, c^2)]
\end{aligned}$$

This also tells us that the lifting map $\mathrm{Hom}(\pi_1(\Sigma), G) \cong Z^{\ell,i}_{\mathrm{flat}}(G)^\tau \to \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)$ can be seen as $(V, c) \to (V, c\mathfrak{r}(V')c^{-1})$ and $(V, d, c) \to (V, d^{-1}c\,\mathfrak{r}(V)\,c^{-1}d, d^{-1}, c^2)$. This precise map will become useful in the following sections.

2.3. **Closed orbits are preserved by the lifting.** Let Γ be a finitely generated group and G is a connected, complex reductive Lie group. Given a representation $\rho : \Gamma \to G$, define $A(\Gamma)$ to be the Zariski closure of $\rho(\Gamma) \subset G$. Then ρ is *semisimple* if and only if the induced representation of $A(\Gamma)$ on $\mathfrak{g}$ (given by composition with the adjoint representation) is semisimple, which occurs if and only if $A(\Gamma)$ is linearly reductive. (See [Ri, Section 3].)

A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is semisimple if and only if the orbit $G \cdot \rho$ is closed in $\mathrm{Hom}(\Gamma, G)$ (this is explained in [Sik, Theorem 30] based on Richardson's proof for free groups in [Ri, Theorem 3.6]). For the case of a non-orientable manifold Σ and its orientable double cover $\tilde{\Sigma}$, pushforward by the covering map $\pi : \tilde{\Sigma} \to \Sigma$ identifies $\pi_1(\tilde{\Sigma})$ with an index 2 subgroup of $\pi_1(\Sigma)$. The main result of this section is Proposition 2.12, a special case of which says that a representation $\rho : \pi_1(\Sigma) \to G$ is semisimple if and only if the lifted representation $\tilde{\rho} : \pi_1(\tilde{\Sigma}) \to G$ is also semisimple. Therefore, the closed orbits in $\mathrm{Hom}(\pi_1(\Sigma), G)$ lift to closed orbits in $\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)$, and every closed orbit in the image of the lifting is the lift of a closed orbit in $\mathrm{Hom}(\pi_1(\Sigma), G)$. We use this in the next section when we define the Betti and de Rham moduli spaces associated to the nonorientable surface.

Proposition 2.12. *Let Γ be a finitely generated group and Γ' an index 2 subgroup. Then a representation ρ of Γ on a finite-dimensional vector space V is semisimple if and only if the induced representation ρ' of Γ' on V is semisimple.*

The proof follows by combining Corollary 2.15 and Lemma 2.17 below. Throughout the proof we use the following results.

(1) For any $c \in \Gamma \setminus \Gamma'$ we have that $c : \Gamma' \to \Gamma \setminus \Gamma'$ is a bijection and $c^2 : \Gamma' \to \Gamma'$.
(2) Γ' is a normal subgroup and so $c\Gamma' c^{-1} = \Gamma'$ and $c^{-1}\Gamma' c = \Gamma'$ for all $c \in \Gamma$.
(3) If W is an irreducible subrepresentation of ρ' then so is cW for all $c \in \Gamma'$ and either $W = cW$ (and hence W is an irreducible subrepresentation of ρ) or $W \cap cW = \{0\}$.

For the remainder of the section we fix a choice of $c \in \Gamma \setminus \Gamma'$ and use ρ to denote a representation of Γ on a finite-dimensional vector space V and ρ' the restriction of ρ to Γ'.

The following elementary example illustrates how ρ' can be the direct sum of two irreducible representations when ρ is irreducible.

Example 2.13. *Consider the groups*

$$\Gamma = \pi_1(\Sigma_1^1) = \left\langle a, b, c \mid [a,b] = c^2 \right\rangle$$
$$\Gamma' = \pi_1(\Sigma_0^2) = \langle x, y, \hat{x}, \hat{y} \mid [x,y][\hat{x},\hat{y}] = \mathrm{id} \rangle\,.$$

The inclusion map $(x, y, \hat{x}, \hat{y}) \mapsto (a, b, cbc^{-1}, cac^{-1})$ *induced by the covering map* $\Sigma_0^2 \to \Sigma_1^1$ *identifies* Γ' *as an index* 2 *subgroup of* Γ. *Fix* $0 < \theta < \pi$ *and consider the representation* $\rho : \Gamma \to \mathsf{SL}(2,\mathbb{C})$ *given by*

$$\rho(a) = \rho(b) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad \rho(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\rho([a,b]) = \mathrm{id} = \rho(c^2)$ *and that* ρ *is irreducible. Then the restriction* $\rho' : \Gamma' \to \mathsf{SL}(2,\mathbb{C})$ *has the form*

$$\rho'(x) = \rho'(y) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \rho'(\hat{x}) = \rho'(\hat{y}) = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

and so ρ' *is the direct sum of two irreducible subrepresentations of dimension* 1. *Note that* $c \in \Gamma$ *interchanges the two subrepresentations.*

The following lemma shows that this is the worst that can happen, i.e. if ρ is irreducible then ρ' can only split into at most two irreducible subrepresentations that are interchanged by c.

Lemma 2.14. *If* ρ *is irreducible then either*

- ρ' *is irreducible, or*
- $V \cong W \oplus cW$, *where* W *and* cW *are irreducible subrepresentations of* ρ'.

Proof. If the first alternative does not hold then there exists a proper non-zero irreducible subrepresentation W of ρ'. Since $W \cap cW$ is a proper subrepresentation of ρ (which is irreducible) then we must have $W \cap cW = \{0\}$. Irreducibility of ρ then implies that $W \oplus cW = V$. $\square$

Corollary 2.15. *If* ρ *is semisimple then* ρ' *is semisimple.*

Proof. Since ρ splits into irreducible subrepresentations and the previous lemma implies that each of these is a semisimple subrepresentation of ρ', then ρ' is a direct sum of semisimple subrepresentations and hence is semisimple. $\square$

Next we show that if ρ' is semisimple then ρ is semisimple. The idea is to show that ρ' splits into subrepresentations which are either: (a) irreducible for both ρ and ρ', or (b) of the form $W \oplus cW$, where W is irreducible for ρ' (which implies that cW is also irreducible for ρ'). Each of these subrepresentations is then either: (a) an irreducible subrepresentation of ρ, or (b) a semisimple subrepresentation of ρ. This second alternative is explained further by the following lemma.

Lemma 2.16. *Let W be an irreducible representation for ρ' and suppose that $W \cap cW = \{0\}$. Note that $W \oplus cW$ is preserved by ρ. If V' is a proper non-zero subrepresentation of ρ on $W \oplus cW$, then*

(1) *$V' \cap W = \{0\}$,*
(2) *V' is irreducible for ρ and the projection maps $p_1 : W \oplus cW \to W$ and $p_2 : W \oplus cW \to cW$ restricted to V' gives isomorphisms of vector spaces $V' \cong W$ and $V' \cong cW$, and*
(3) *there is an irreducible subrepresentation V'' of ρ such that $W \oplus cW = V' \oplus V''$.*

Proof. (1) Note that V' is also a subrepresentation for ρ', and therefore so is $V' \cap W$. Since W is irreducible for ρ', then either we have $V' \cap W = \{0\}$ or $V' \cap W = W$. In the second case, since c preserves V', then we must have $V' \cap cW = cW$ also, which means that $V' = W \oplus cW$, contradicting the assumption that V' is a proper subrepresentation of $W \oplus cW$. Therefore $V' \cap W = \{0\}$. The same argument shows that $V' \cap cW = \{0\}$.

(2) Let $V_1 \subset V'$ be an irreducible subrepresentation for ρ. We claim that $V_1 = V'$. Since V' is a proper subspace of $W \oplus cW$ and V_1 is a subrepresentation of ρ', then by [Pr, Corollary 2, p155] we must have $V' \supset V_1 \cong W$ or $V' \supset V_1 \cong cW$ (or both), with the isomorphism given by the projections p_1 or p_2 respectively. If $V_1 \neq V'$ then $\dim V' > \dim W = \dim cW$, and therefore $V' \cap cW \neq \{0\}$ and $V' \cap W \neq \{0\}$ for dimensional reasons, contradicting the result above. Therefore $V' = V_1$ and so V' is irreducible for ρ. Moreover, since $V' \cap W = \{0\} = V' \cap cW$, then [Pr, Corollary 2, p155] again shows that the projection maps $p_1 : V' \to W$ and $p_2 : V' \to cW$ are both isomorphisms.

(3) Since $V' \subset W \oplus cW$, then every $v \in V'$ can be written uniquely as $v = w_1 + cw_2$, where $w_1, w_2 \in W$. Conversely, since the projections p_1 and p_2 are isomorphisms, then for every $w_1 \in W$ there exists a unique $w_2 \in W$ such that $w_1 + cw_2 \in V'$. We define the space V'' as

$$V'' := \{w_1 - cw_2 \,:\, w_1, w_2 \in W \,\text{and}\, w_1 + cw_2 \in V'\}\,.$$

Clearly $\dim V'' = \dim V'$. To complete the proof, we need to show that (a) $V' \oplus V'' = W \oplus cW$ and (b) V'' is preserved by ρ, i.e. $\Gamma' V'' = V''$ and $cV'' = V''$.

Suppose $w_1 - cw_2 \in V' \cap V''$. Then $w_1 + cw_2 \in V'$ and so $2w_1 = w_1 - cw_2 + w_1 + cw_2 \in V'$ also. Similarly, $2cw_2 \in V'$. Since $W \cap V' = \{0\}$ and $cW \cap V' = \{0\}$, then we must have $w_1 = 0$ and $w_2 = 0$. Therefore $V' \cap V'' = \{0\}$. Since $\dim V' = \dim V'' = \dim W = \dim cW$ and $V' \oplus V''$ is contained in $W \oplus cW$, then we must have $V' \oplus V'' = W \oplus cW$.

To see that $cV'' = V''$, consider a general element $w_1 - cw_2 \in V''$. Then $w_1 + cw_2 \in V'$ and so $c^2 w_2 + cw_1 \in V'$ since c preserves V'. Therefore, since $c^2 \in \Gamma'$ implies $c^2 w_2 \in W$, we have $c^2 w_2 - cw_1 = -c(w_1 - cw_2) \in V''$.

To see that $\Gamma' V'' = V''$, consider a general element $w_1 - cw_2 \in V''$. Then $w_1 + cw_2 \in V'$ and, since $\Gamma' V' = V'$, we have $\gamma w_1 + \gamma c w_2 = \gamma w_1 + c(c^{-1}\gamma c w_2) \in V'$ for any $\gamma \in \Gamma'$. Since $\Gamma' W = W$ then $c^{-1}\gamma c w_2 \in W$ and so

$$\gamma(w_1 - cw_2) = \gamma w_1 - \gamma c w_2 = \gamma w_1 - c(c^{-1}\gamma c w_2) \in V''.$$

Therefore we have constructed a subrepresentation V'' of ρ such that $V' \oplus V'' = W \oplus cW$. Replacing V' by V'' in part (2) of the lemma shows that V'' is irreducible, which completes the proof. $\square$

Lemma 2.17. *Suppose that ρ' is semisimple. Then ρ is also semisimple.*

Proof. The proof uses induction to show that V splits into irreducible subrepresentations for ρ. First, since there is an irreducible subrepresentation V_1 for ρ and ρ' is semisimple, then $V \cong V_1 \oplus V'$, where V' is a subrepresentation of ρ'. This is the base case of the induction.

Now suppose that there exists a direct sum decomposition

$$V \cong V_k' \oplus \bigoplus_{i=1}^{k} V_i,$$

where each V_i is an irreducible subrepresentation for ρ and V_k' is a subrepresentation for ρ'. To complete the inductive step, we aim to show that there is an irreducible subrepresentation V_{k+1} of ρ such that

$$V_{k+1} \cap \bigoplus_{i=1}^{k} V_i = \{0\}.$$

Semisimplicity of ρ' then implies the existence of a subrepresentation V_{k+1}' such that

$$V \cong V_{k+1}' \oplus \bigoplus_{i=1}^{k+1} V_i$$

and the fact that V is finite-dimensional shows that this process must eventually terminate.

The proof of the inductive step is as follows. Since ρ' is semisimple, then there is an irreducible subrepresentation $W \subset V_k'$ for ρ' such that $V_k' \cong W \oplus V_k''$. Since W is irreducible for ρ' then cW is also irreducible for ρ' and so either $W \cap cW = W$ or $W \cap cW = \{0\}$. In the first case, W is an irreducible subrepresentation of ρ, and so we can set $V_{k+1}' = W$. When $W \cap cW = \{0\}$ then $W \oplus cW$ is a subrepresentation for ρ' and there are two further cases:

- $W \oplus cW$ is irreducible for ρ, or

- there is a proper subrepresentation $V' \subset W \oplus cW$ for ρ.

(Note that we may not have $cW \subset V'_k$, but we do have $cW \cap \oplus_{i=1}^k V_i = \{0\}$.)

In the first case, we immediately have that $(W \oplus cW) \cap \oplus_{i=1}^k V_i = \{0\}$, since the intersection is a subrepresentation for ρ and it cannot be all of $W \oplus cW$ since $W \subset V'_k$. Therefore we can define $V'_{k+1} = W \oplus cW$.

In the second case, we can invoke the previous lemma to show that $W \oplus cW = V' \oplus V''$, where V' and V'' are irreducible subrepresentations for ρ. If $V' \cap \oplus_{i=1}^k V_i = \{0\}$ then we can set $V_{k+1} = V'$. If $V' \cap \oplus_{i=1}^k V_i \neq \{0\}$ then we have $V'' \cap \oplus_{i=1}^k V_i = \{0\}$ since the intersection is a subrepresentation of ρ on V'' and so must be either $\{0\}$ or V'', but we can exclude the latter case since it would imply that $W \subset V' \oplus V'' \subset \oplus_{i=1}^k V_i$, contradicting $W \cap \oplus_{i=1}^k V_i = \{0\}$. In this case we can set $V'_{k+1} = V''$, which completes the inductive step.

Therefore V is the direct sum of irreducible subrepresentations for ρ, and so ρ must be semisimple also. □

2.4. **Moduli spaces.** In this section we define the moduli spaces that will be studied in the rest of the paper. These are the nonorientable analogs of the Betti, de Rham and Dolbeault moduli spaces from nonabelian Hodge theory (cf. [S2]).

Definition 2.18. *The* Betti moduli space *is the categorical quotient*

$$\mathcal{M}^{Betti}(\Sigma) \stackrel{\text{def}}{=} \operatorname{Hom}(\pi_1(\Sigma), G)/\!\!/G, \tag{2.4}$$

where the quotient $/\!\!/$ means that we identify orbits whose closures intersect (this is an equivalence relation by [Na2, Theorem 4, p19]*, see also* [Na1]*).*

Recall from [G2] that a *reductive representation* $\rho : \Gamma \to G$ is a representation for which the Zariski closure of the image is a reductive subgroup of G. This is equivalent to the notion of a *semisimple representation* $\rho : \Gamma \to G$: a representation for which the induced adjoint representation of Γ on $\mathfrak{g}$ is semisimple (see [Ri], [BGG, p205]).

Remark 2.19. *Since the closure of each orbit contains a unique closed orbit (see* [Na1, Sec. 8]*) and the closed orbits are precisely the semisimple representations (see* [Sik, Theorem 30]*), then we have*

$$\mathcal{M}^{Betti}(\Sigma) = \operatorname{Hom}(\pi_1(\Sigma), G)^{red}/G.$$

Definition 2.20. *The* de Rham moduli space *is the quotient*

$$\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}}) \stackrel{\text{def}}{=} \mathcal{A}^{\text{flat}}(P^{\mathbb{C}})/\!\!/\mathcal{G}^{\mathbb{C}}, \tag{2.5}$$

where the quotient $/\!\!/$ identifies orbits whose closures intersect.

Remark 2.21. *For the same reasons as above, there is an equivalent definition using the reductive flat connections on $P^{\mathbb{C}}$.*

$$\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}}) = \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}})/\mathcal{G}^{\mathbb{C}}.$$

There is a well-known correspondence between gauge-equivalence classes of flat connections and representations of $\pi_1(\Sigma)$ (see for example [Ko, Prop. 2.6]). This gives us

$$\mathcal{M}^{Betti}(\Sigma) \cong \bigcup_{[P^{\mathbb{C}}]} \mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}}), \tag{2.6}$$

where we take the union over all smooth equivalence classes of $K^{\mathbb{C}}$ bundles on Σ that admit a flat connection.

We can also define moduli spaces of τ-invariant flat connections on $\tilde{P}^{\mathbb{C}}$.

Definition 2.22.

$$\mathcal{M}^{dR}_{\tau}(\tilde{\Sigma}, \tilde{P}^{\mathbb{C}}) := \mathcal{A}^{\text{flat}}_{\tau}(\tilde{P}^{\mathbb{C}}) /\!\!/ \tilde{\mathcal{G}}^{\mathbb{C}}_{\tau} \cong \mathcal{A}^{\text{flat,red}}_{\tau}(\tilde{P}^{\mathbb{C}})/\tilde{\mathcal{G}}^{\mathbb{C}}_{\tau}. \tag{2.7}$$

From the discussion in Section 2.3, we have the following lemma (in the language of flat connections).

Lemma 2.23. *The restriction of $\pi^* : \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \to \mathcal{A}^{\text{flat}}_{\tau}(\tilde{P}^{\mathbb{C}})$ to $\mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}})$ is a homeomorphism $\pi^* : \mathcal{A}^{\text{flat,red}}(P^{\mathbb{C}}) \to \mathcal{A}^{\text{flat,red}}_{\tau}(\tilde{P}^{\mathbb{C}})$.*

This immediately gives us

Lemma 2.24.

$$\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}}) \cong \mathcal{M}^{dR}_{\tau}(\tilde{\Sigma}, \tilde{P}^{\mathbb{C}}). \tag{2.8}$$

In Section 5 we show that the smooth part of $\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}})$ has a symplectic structure. It is then natural to ask for criteria determining whether a representation ρ (or equivalently, a flat connection D) corresponds to a smooth point in the moduli space. This question is studied in the paper [HW], however it is worth mentioning in this paper one difference between nonorientable surfaces and the case of a compact Riemann surface studied by Goldman in [G1].

Recall that a *simple representation* (in the orientable case) is one that has isotropy equal to the centre of G. Goldman shows in [G1] that the simple representations correspond to the smooth points in the Betti moduli space. For a representation $\rho : \pi_1(\Sigma) \to G$ to lift to a simple representation in $\mathcal{M}^{Betti}(\tilde{\Sigma})$ it is necessary that ρ have isotropy group $Z(G)$. Example 2.13 in the previous section shows that this is not sufficient, i.e. a representation with isotropy $Z(G)$ may lift to a representation with non-trivial isotropy.

Following Goldman's notation from [G1], let $\mathcal{M}^{Betti}(\Sigma)^{--}$ denote the subset of reductive representations $\pi_1(\Sigma) \to G$ with isotropy $Z(G)$. The results of [HW] show that obstructions appear in the deformation theory of $\mathcal{M}^{Betti}(\Sigma)^{--}$ whenever an irreducible representation $\rho : \pi_1(\Sigma) \to G$ lifts to a representation $\tilde{\rho} : \pi_1(\tilde{\Sigma}) \to G$ that is the direct sum of two irreducible subrepresentations (cf. Example 2.13 and Lemma 2.14 in the previous section). Therefore $\mathcal{M}^{Betti}(\Sigma)^{--}$ is not necessarily smooth and we give an example in [HW] that shows in fact that singular points do exist, and that the same is true for the space of flat reductive connections with isotropy $Z(G)$. Therefore, in contrast to the case of a compact Riemann surface (see [G1]), the first inclusion in (2.9) below is not always an equality.

$$\mathcal{M}^{Betti}(\Sigma)^{\text{smooth}} \subset \mathcal{M}^{Betti}(\Sigma)^{--} \subset \mathcal{M}^{Betti}(\Sigma) \tag{2.9}$$

Returning to the definition of a Higgs bundle, recall that given a reduction of structure group on $P^{\mathbb{C}}$ from G to K, a connection D on $P^{\mathbb{C}}$ decomposes as $D = d_A - i\psi$, where d_A is a connection on P and $\psi \in \Omega^1(\mathrm{ad}P)$. The gauge group on P is denoted $\mathcal{G}$. Using the operator d_A^* from Definition 2.3, we can write down the following equation

$$d_A^* \psi = 0. \tag{2.10}$$

If we assume that the double cover has a complex structure, then we can decompose the pullback of d_A and ψ into $(0,1)$ and $(1,0)$ parts:

$$\begin{aligned} \pi^*(d_A) &= \bar{\partial}_A + \partial_A \\ \pi^*(\psi) &= i\phi + i\phi^*. \end{aligned}$$

We can now define a Higgs bundle.

Definition 2.25. *Suppose that $\tilde{\Sigma}$ is a complex manifold. Fix a reduction of structure group from $P^{\mathbb{C}}$ to P. A connection $D = d_A - i\psi \in \mathcal{A}(P^{\mathbb{C}})$ is a* Higgs bundle *if and only if the following conditions are satisfied.*

- $d_A \psi = 0$,
- $d_A^* \psi = 0$, *and*
- *the associated operators $\bar{\partial}_A$ and ϕ satisfy $(\bar{\partial}_A)^2 = 0$ and $\phi \wedge \phi = 0$.*

The space of Higgs bundles *is denoted*

$$\mathcal{A}^{Higgs}(P) \stackrel{\text{def}}{=} \left\{ D = d_A - i\psi \in \mathcal{A}(P^{\mathbb{C}}) \, : \, d_A\psi = 0, d_A^*\psi = 0, (\bar{\partial}_A)^2 = 0, \phi \wedge \phi = 0 \right\} \tag{2.11}$$

and the space of solutions to Hitchin's equations *is*

$$\mathcal{A}^{Hitchin}(P) \stackrel{\text{def}}{=} \mathcal{A}^{Higgs}(P) \cap \mathcal{A}^{\text{flat}}(P^{\mathbb{C}}) \tag{2.12}$$

The moduli space of solutions to Hitchin's equations *is*

$$\mathcal{M}^{Hitchin}(\Sigma, P) \stackrel{\text{def}}{=} \mathcal{A}^{Hitchin}(P)/\mathcal{G}. \tag{2.13}$$

The following lemma appears in [S2, Lemma 1.1] for Higgs bundles over compact Kähler manifolds. It shows that if the connection is flat and the bundle is harmonic, then the conditions $(\bar{\partial}_A)^2 = 0$ and $\phi \wedge \phi = 0$ are automatically satisfied.

Lemma 2.26 (Siu, Sampson, Corlette, Deligne)**.** *Suppose that $\tilde{\Sigma}$ is compact and Kähler. Let $D = d_A - i\psi$ be a flat connection on $\tilde{P}^{\mathbb{C}} \to \tilde{\Sigma}$ and let $d_A = \bar{\partial}_A + \partial_A$ and $\psi = i\phi + i\phi^*$ be the decomposition into* $(0,1)$ *and* $(1,0)$ *parts. Then $d_A^* \psi = 0$ implies that $(\bar{\partial}_A)^2 = 0$ and $\phi \wedge \phi = 0$.*

As a consequence, we can omit the conditions $(\bar{\partial}_A)^2 = 0$ and $\phi \wedge \phi = 0$ in the definition of $\mathcal{A}^{Hitchin}(P)$ when $\tilde{\Sigma}$ is compact and Kähler (but not in the definition of $\mathcal{A}^{Higgs}(P)$, since the connection may not be flat). In particular, for the results of Section 3, it is sufficient to find a solution of $d_A^* \psi = 0$ on the space $\mathcal{A}^{flat}(P^{\mathbb{C}})$.

Remark 2.27. *The definition of $\mathcal{A}^{Hitchin}(P)$ (and hence $\mathcal{M}^{Hitchin}(\Sigma, P)$) given above depends on the choice of reduction of structure group. A different choice will define a different subset of $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ satisfying* (2.12)*. Theorem 3.1 in the next section shows that, given a fixed choice of reduction of structure group, the $\mathcal{G}^{\mathbb{C}}$-orbit of each reductive flat connection always contains a unique $\mathcal{G}$ orbit consisting of flat connections satisfying* (2.10)*. Equivalently, given a fixed reductive flat connection D, there is a unique reduction of structure group such that $D \in \mathcal{A}^{Higgs}(P) \cap \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$.*

Remark 2.28. *When the base manifold is compact and Kähler then a theorem of Hitchin and Simpson (see* [Hi1] *and* [S1]*) identifies $\mathcal{M}^{Hitchin}(P)$ with a moduli space of stable objects in $\mathcal{A}^{Higgs}(P)$, which Simpson calls the* Dolbeault moduli space *in* [S2]*. In this paper we define $\mathcal{M}^{Hitchin}(P)$ directly, and so we use the terminology* Hitchin moduli space *instead. Since Σ is not orientable then $\mathcal{M}^{Hitchin}(P)$ is not hyperkähler, and so we avoid the terminology* hyperkähler quotient*.*

3. The Donaldson-Corlette correspondence

Recall the setup from the previous section: Σ is a compact, closed nonorientable manifold, K is a compact, connected semisimple Lie group and $P \to \Sigma$ is a principal K-bundle. Let G and $P^{\mathbb{C}}$ denote the corresponding complexifications, let $\mathcal{A}(P^{\mathbb{C}}) \cong \Omega^1(\Sigma, \mathrm{ad} P^{\mathbb{C}}) \cong T^* \mathcal{A}(P)$ denote the space of connections on $P^{\mathbb{C}}$ and let $\mathcal{A}^{flat}(P^{\mathbb{C}})$ denote the subset of flat connections. In this section we also assume that the orientable double cover of Σ admits a Kähler structure.

The goal of this section is to prove a correspondence between the moduli spaces $\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}})$ and $\mathcal{M}^{Hitchin}(\Sigma, P)$, and hence with a component of $\mathcal{M}^{Betti}(\Sigma)$ (see (2.6)).

It is well known that $\mathcal{M}^{Betti}(\Sigma) \cong \bigcup_{[P^{\mathbb{C}}]} \mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}})$. The following theorem relates $\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}})$ and $\mathcal{M}^{Hitchin}(\Sigma, P)$.

Theorem 3.1. *Let Σ be a compact non-orientable manifold whose orientable double cover $\tilde{\Sigma}$ admits a Kähler structure. For every reductive flat connection D on $P^{\mathbb{C}}$, there is a gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$ (unique up to the action of $\mathcal{G}$) such that $g \cdot D = d_A - i\psi$ solves* (2.10). *Equivalently, there is a unique reduction of structure group from $P^{\mathbb{C}}$ to P admitting a solution to* (2.10). *Moreover, when $\dim_{\mathbb{R}} \tilde{\Sigma} = 2$ there is a continuous $\mathcal{G}$-equivariant deformation retract from $\mathcal{A}^{flat}(P^{\mathbb{C}})$ onto $\mathcal{A}^{Hitchin}(P)$.*

As a consequence, we have

$$\mathcal{M}^{dR}(\Sigma, P^{\mathbb{C}}) \cong \mathcal{M}^{Hitchin}(\Sigma, P).$$

This is the nonorientable analog of the Donaldson-Corlette theorem from [D2] and [C]. The proof involves studying the gradient flow of the harmonic map energy functional on the space of metrics on $P^{\mathbb{C}}$. Alternately, one can fix the metric and analyse the negative gradient flow of the moment map functional

$$\begin{aligned} f : T^*\mathcal{A}(P) &\to \mathbb{R} \\ d_A - i\psi &\mapsto \|d_A^* \psi\|^2. \end{aligned} \tag{3.1}$$

on the space $\mathcal{A}^{flat}$. The change of metric given by the solution to the harmonic map flow lifts to a complex gauge transformation that generates the solutions to the gradient flow of (3.1).

Rather than deriving all of the results from scratch, instead the strategy of the proof is to use Corlette's results on long-time existence and convergence for the flow on an orientable manifold. In Section 3.1 we give necessary and sufficient conditions for the flow on the nonorientable manifold to lift to a solution of Corlette's flow on the orientable double cover. Essentially the problem reduces to showing short-time existence for the flow on the nonorientable manifold. In Section 3.2 we show that these conditions are satisfied, and the proof of Theorem 3.1 then follows from the results of [C]. The statement about the deformation retraction follows from [DWW, Proposition 2.1].

3.1. **Sufficient conditions for the flows to coincide.** In this section we prove the following general result that applies directly to the case at hand: studying the flow on the space of flat connections on a non-orientable manifold via the induced flow on the subset of the space of flat connections that is invariant under the orientation-reversing involution.

Proposition 3.2. *Let N and M be manifolds, $i : N \hookrightarrow M$ an embedding, $S_N \subset N$ and $S_M \subset M$ closed subsets (possibly singular), and X and Y vector fields on M and N respectively such that*

(1) *$i(S_N) \subset S_M$ and $i(S_N)$ is closed in S_M,*
(2) *$di_y(Y(y)) = X(i(y))$ for all $y \in N$,*

(3) *for each* $y \in S_N$ *there exists* $\varepsilon > 0$ *and a integral curve* $z_N : [0, \varepsilon) \to S_N$ *for* Y *on* N *with* $z_N(0) = y$,
(4) *for each* $x \in S_M$ *there exists a unique integral curve* $z_M : [0, \infty) \to S_M$ *for* X *on* M *with* $z_M(0) = x$ *such that* $\lim_{t \to \infty} z_M(t)$ *exists and is contained in* S_M.

Then for each $y \in S_N$ *there is a unique integral curve* $z_N : [0, \infty) \to S_N$ *for* Y *on* N *with initial condition* y *such that* $\lim_{t \to \infty} z_N(t)$ *exists and is contained in* S_N. *Moreover,* $i \circ z_N$ *coincides with the integral curve* $z_M : [0, \infty) \to S_M$ *with initial condition* $z(0) = i(y)$.

Remark 3.3. (1) *Note that the vector fields* X *and* Y *and the associated integral curves are defined with respect to the manifolds* M *and* N, *not the subsets* S_N *and* S_M *(which may not be manifolds, and therefore the usual definitions of tangent spaces, vector fields, etc. don't make sense). The assumption is that each integral curve with initial conditions in* S_M *(resp.* S_N*) remains in the subset* S_M *(resp.* S_N*).*
(2) *It is not necessary to assume existence of solutions that have initial conditions in* $M \setminus S_M$ *or* $N \setminus S_N$.
(3) *There is no assumption of compactness of* N *or* M, *just that the integral curves on* S_M *converge in* S_M *and that* $i(S_N)$ *is closed.*
(4) *The result applies regardless of whether the spaces are finite or infinite-dimensional.*

Proof. The first two conditions show that $i(z_N)$ is an integral curve for X on M with initial condition $i(y)$. Let $z_M(t)$ be the integral curve for X on M with initial condition $i(y) \in S_M$. Uniqueness of integral curves on M with initial conditions in S_M shows that $i(z_N(t)) = z_M(t)$ for all $t \in [0, \varepsilon)$ and therefore $z_N(t) = i^{-1}(z_M(t))$ is the unique integral curve for Y on N with $z_N(0) = y$.

Since $z_M(\varepsilon) \in S_M$, the map i is an embedding, and the image of the restriction of i to S_N is closed, then $\lim_{t \to \varepsilon} z_N(t)$ exists and is equal to $i^{-1}(z_M(\varepsilon))$. Therefore we can use local existence again to extend the integral curve $z_N(t)$ past $t = \varepsilon$ and a repeated application of this argument shows that $z_N(t)$ exists for all time and that $i(z_N(t))$ coincides with $z_M(t)$. The sets $S_M \subset M$ and $i(S_N) \subset S_M$ are closed, and so $i(z_N(t))$ converges to a point in $i(S_N)$. Therefore $\lim_{t \to \infty} z_N(t)$ exists and is contained in S_N. □

Clearly the first two assumptions of the proposition are necessary. The following two elementary examples show that the final two hypotheses of the proposition are also necessary. In the first example we remove the hypothesis that flow lines on M are unique.

Example 3.4. *Let $M = \mathbb{R}$ and $N = \{0\} \subset \mathbb{R}$. Let $X(x) = x^{\frac{1}{3}}$. Then there are two solutions with initial condition $x(0) = 0$: $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ and $x(t) = 0$ for all $t \in \mathbb{R}$. The second solution is contained in the submanifold N, however the first one is not. Therefore there exists an integral curve on M with initial conditions in N that does not remain in N.*

In the next example we remove the hypothesis of *a priori* existence for integral curves on N. We still have existence and uniqueness of integral curves on M. This example also shows that existence of integral curves for X on M doesn't imply existence of integral curves for the restriction of X to N, even if X is always tangent to N.

Example 3.5. *Let $M \in \mathbb{R}^2$ and $N = \{(x, y) \in M \,:\, y = 0\} \cong \mathbb{R}$. Define*

$$X(x, y) = \begin{cases} (1, 2x) & y = x^2 \\ (0, 0) & y \neq x^2 \end{cases}$$

Note that $X(0, 0) = (1, 0)$ and $X(x, 0) = (0, 0)$ if $x \neq 0$. Therefore $X(x, y)$ is tangent to N for all $(x, y) \in N$. Note also that there is no integral curve with initial condition $(0, 0)$ for the restriction of X to N.

The integral curve through any (x_0, y_0) such that $y_0 = x_0^2$ is given by $(x(t), y(t)) = (t - x_0, (t - x_0)^2)$. If $y_0 \neq x_0^2$ then the integral curve is $(x(t), y(t)) = (x_0, y_0)$.

Now consider the initial condition $(0, 0) \in N$. The unique integral curve through $(0, 0)$ is (t, t^2), which does not remain in N.

3.2. **Moment map flow on the space of flat connections over a nonorientable manifold.** In this section we show that the conditions of Proposition 3.2 are satisfied for the gradient flow of f. The space S_N corresponds to the space of flat connections on the nonorientable manifold, S_M corresponds to the space of flat connections on the orientable double cover (which we assume to be Kähler), and i is the inclusion map corresponding to pullback $p^* : T^*\mathcal{A} \to T^*\tilde{\mathcal{A}}$ by the projection $p : \tilde{\Sigma} \to \Sigma$.

The pullback of a flat connection by p is also a flat connection and $p^*(\mathcal{A}^{flat})$ is closed in $\tilde{\mathcal{A}}^{flat}$, so the first condition of Proposition 3.2 is satisfied. Corlette's results on the long time existence and convergence of the gradient flow of (3.1) on the orientable double cover show that the fourth condition is satisfied. (See also [DWW, Proposition 2.1] for a proof that the gradient flow defines a deformation retract.) The following proposition shows that the second and third conditions are also satisfied.

Proposition 3.6. *Fix a reduction of structure group from $P^{\mathbb{C}}$ to P. Then*

1. *Let $\tilde{f}$ denote the energy functional (3.1) on the orientable double cover $\tilde{\Sigma}$. Then $p^*(\operatorname{grad} f(D)) = \operatorname{grad} \tilde{f}(p^*(D))$ for all $D \in \mathcal{A}^{flat}$.*

(2) *For any* $D = d_A - i\psi \in \mathcal{A}^{flat}$, *there exists* $\varepsilon > 0$ *such that the gradient flow of the energy function* f *from* (3.1) *exists on the interval* $[0, \varepsilon)$.

Assuming this result, the proof of Theorem 3.1 then follows from Proposition 3.2 and the results of [C].

The proof of Proposition 3.6 occupies the rest of the section. We derive the flow equations explicitly and show that short-time existence still holds for the τ^*-invariant subset. More precisely, we show that the induced parabolic equation on the space of metrics is τ^*-invariant. Corlette omits the derivation of the flow equations since it is similar in spirit to the approach of [D1], however we include the details here since the explicit description is essential for the proof of Theorem 3.1.

Recall the basic setup. We have $d_A \in \mathcal{A}$ and $\psi \in \Omega^1(\mathrm{ad}(P))$. The energy functional is

$$f(d_A, \psi) = \|d_A^* \psi\|^2 = \int_X |d_A^* \psi|^2 \,\mathrm{dvol}\,.$$

Lemma 3.7.

$$\mathrm{grad}\, f(d_A, \psi) = ([d_A^* \psi, \psi], d_A d_A^* \psi)\,.$$

Proof. The proof is just a calculation. First we compute the derivative

$$df_{(d_A, \psi)}(a, \varphi) = 2\Re \left\langle - * [a, *\psi] + d_A^* \varphi, d_A^* \psi \right\rangle .$$

We also have

$$\left\langle d_A^* \psi, - * [a, *\psi] \right\rangle = \left\langle a, [d_A^* \psi, \psi] \right\rangle .$$

Therefore

$$\begin{aligned} df_{(d_A, \psi)}(a, \varphi) &= 2\Re \left\langle - * [a, *\psi], d_A^* \psi \right\rangle + \left\langle d_A^* \varphi, d_A^* \psi \right\rangle \\ &= 2\Re \left\langle a, [d_A^* \psi, \psi] \right\rangle + \left\langle \varphi, d_A d_A^* \psi \right\rangle , \end{aligned}$$

and so the gradient is

$$\mathrm{grad}\, f(d_A, \psi) = ([d_A^* \psi, \psi], d_A d_A^* \psi)\,.$$

□

As a consequence, the downwards gradient flow equations are

$$\frac{\partial A}{\partial t} = -[d_A^* \psi, \psi] \tag{3.2}$$

$$\frac{\partial \psi}{\partial t} = -d_A d_A^* \psi. \tag{3.3}$$

In particular, we see that given $D = d_A - i\psi \in \mathcal{A}^{flat}$, then $(\tilde{d_A}, \tilde{\psi}) = p^*(d_A, \psi)$ satisfies

$$[\tilde{d_A}^* \tilde{\psi}, \tilde{\psi}] = p^*[d_A^* \psi, \psi], \quad \tilde{d_A} \tilde{d_A}^* \tilde{\psi} = p^*(d_A d_A^* \psi),$$

which proves the first statement of Proposition 3.6. The rest of the section is devoted to proving the second statement, i.e. proving short-time existence for the gradient flow equations (3.2) and (3.3).

Next, we show that the solutions to the flow equations are generated infinitesimally by the complex gauge group. The first step is to compute the infinitesimal action.

Let $D = d_A - i\psi$ denote the connection on $P^{\mathbb{C}}$ formed from the unitary connection and the Higgs field. The action of $g \in \mathcal{G}^{\mathbb{C}}$ is given by

$$g \cdot D = gDg^{-1},$$

which acts on a section s by $(g \cdot D)(s) = gD(g^{-1}s)$.

Therefore the action on d_A is given by taking the skew Hermitian part of $g(d_A - i\psi)g^{-1}$ and the action on ψ is given by taking the Hermitian part. The infinitesimal action of $u \in \operatorname{Lie} \mathcal{G}^{\mathbb{C}}$ is

$$\rho_D(u) = -Du = -(d_A u - i[\psi, u])$$

The connection part of the infinitesimal action of $u \in \operatorname{Lie} \mathcal{G}^{\mathbb{C}}$ is then

$$-\frac{1}{2}\left(d_A u - [i\psi, u] - (d_A u - [i\psi, u])^*\right)$$

and the Higgs field part of the infinitesimal action is given by taking the Hermitian part

$$-\frac{1}{2}\left(d_A u - [i\psi, u] + (d_A u - [i\psi, u])^*\right).$$

Note that this is the infinitesimal action on $-i\psi$ (which is the Hermitian part of $d_A - i\psi$). The infinitesimal action on ψ is

$$-\frac{1}{2} i \left(d_A u - [i\psi, u] + (d_A u - [i\psi, u])^*\right).$$

Therefore, the infinitesimal action of $u \in \operatorname{Lie} \mathcal{G}^{\mathbb{C}}$ at (d_A, ψ) is

$$\rho_{(d_A,\psi)}(u) = \begin{pmatrix} -\frac{1}{2}\left(d_A u - [i\psi, u] - (d_A u - [i\psi, u])^*\right) \\ -\frac{1}{2} i \left(d_A u - [i\psi, u] + (d_A u - [i\psi, u])^*\right) \end{pmatrix}. \tag{3.4}$$

Now let $u(d_A, \psi) = i d_A^* \psi$. Then $u^* = u$ and we have

$$\rho_{(d_A,\psi)}(u) = \begin{pmatrix} [i\psi, u] \\ -i d_A u \end{pmatrix} = \begin{pmatrix} -[\psi, d_A^* \psi] \\ d_A d_A^* \psi \end{pmatrix} = \begin{pmatrix} [d_A^* \psi, \psi] \\ d_A d_A^* \psi \end{pmatrix}. \tag{3.5}$$

Equation (3.5) shows that we can write the downwards gradient flow equations (3.2) and (3.3) using the infinitesimal action of the complex gauge group

$$\left(\frac{\partial A}{\partial t}, \frac{\partial \psi}{\partial t}\right) = -\rho_{(d_A,\psi)}(u(d_A, \psi)). \tag{3.6}$$

The following calculation shows that the flow is generated by the action of $\mathcal{G}^{\mathbb{C}}$. Fix (A_0, ψ_0) and let $g(t)$ be the solution to the equation

$$\frac{\partial g}{\partial t} g^{-1} = -u(g(t) \cdot A_0, g(t) \cdot \psi_0), \tag{3.7}$$

Let $(A(t), \psi(t)) = g(t) \cdot (A_0, \psi_0)$. Then

$$\begin{aligned} \frac{\partial}{\partial t} \left(A(t), \psi(t) \right) &= \rho_{(A(t), \psi(t))} \left(\frac{\partial g}{\partial t} g(t)^{-1} \right) \\ &= -\rho_{(A(t),\psi(t))}(u(A(t), \psi(t))). \end{aligned}$$

Equation (3.6) then shows that $g(t) \cdot (A_0, \psi_0)$ gives a solution to the gradient flow equations.

It still remains to prove that solutions to (3.7) exist. The goal is to rewrite this equation in terms of a change of metric, which will then satisfy a parabolic equation (cf. [D1] for the Yang-Mills flow, where $\frac{\partial g}{\partial t}$ satisfies the equation on [D1, p7] and the change of metric satisfies the parabolic equation (B) on [D1, p13]). With this in mind, define the change of metric $h(t) = g(t)^* g(t)$. We then have

$$\begin{aligned} \frac{\partial h}{\partial t} &= \frac{\partial g^*}{\partial t} g + g^* \frac{\partial g}{\partial t} \\ &= g^* \left(\left(\frac{\partial g}{\partial t} g^{-1} \right)^* + \frac{\partial g}{\partial t} g^{-1} \right) g \\ &= -h g^{-1} u(g \cdot A_0, g \cdot \psi_0) g. \end{aligned} \tag{3.8}$$

Next we compute $g^{-1} u(g \cdot A_0, g \cdot \psi_0) g$ (see Lemma 3.11). Consider the pullback of (d_A, ψ) to the orientable double cover. To simplify notation, in the rest of the section we also use (d_A, ψ) to denote this pullback. The meaning will be clear from the context.

Using the complex structure on the orientable double cover, split the connection and the Higgs field into $(0,1)$ and $(1,0)$ parts

$$d_A = \bar{\partial}_A + \partial_A, \quad \psi = i\phi + i\phi^*$$

and define operators

$$\begin{aligned} L_K^{(1)} &= \bar{\partial}_A + \phi^* + \partial_A - \phi \\ L_K^{(2)} &= \bar{\partial}_A - \phi^* - \partial_A - \phi. \end{aligned}$$

Note that *a priori* these are defined on the orientable double cover, since we use the complex structure of the underlying manifold in the definition. We will see that the final result for $u(g \cdot A, g \cdot \psi)$ is τ-invariant, and therefore descends to the non-orientable manifold.

Lemma 3.8. *Let $d_A = \bar{\partial}_A + \partial_A$ and $\psi = i(\phi + \phi^*)$. Then*

$$u(d_A, \psi) = \frac{1}{2} i \Lambda \left(L_K^{(1)} L_K^{(1)} + L_K^{(2)} L_K^{(2)} \right) \tag{3.9}$$

Proof. The proof is just a calculation. Firstly, the $(1,1)$ component of $L_K^{(1)}L_K^{(1)} + L_K^{(2)}L_K^{(2)}$ is

$$\begin{aligned}
(\bar\partial_A + \phi^* + \partial_A - \phi)&(\bar\partial_A + \phi^* + \partial_A - \phi)^{(1,1)} \\
&+ (\bar\partial_A - \phi^* - \partial_A - \phi)(\bar\partial_A - \phi^* - \partial_A - \phi)^{(1,1)} \\
&= F_A - [\phi, \phi^*] + \partial_A \phi^* - \bar\partial_A \phi - F_A + [\phi, \phi^*] + \partial_A \phi^* - \bar\partial_A \phi \\
&= 2\partial_A \phi^* - 2\bar\partial_A \phi.
\end{aligned}$$

Using the Kähler identities $\partial_A^* = i[\Lambda, \bar\partial_A]$ and $\bar\partial_A^* = -i[\Lambda, \partial_A]$ leads to

$$\begin{aligned}
\frac{1}{2} i\Lambda \left(L_K^{(1)} L_K^{(1)} + L_K^{(2)} L_K^{(2)} \right) &= i\Lambda \partial_A \phi^* - i\Lambda \bar\partial_A \phi \\
&= -\bar\partial_A^* \phi^* - \partial_A^* \phi \\
&= -d_A^*(\phi + \phi^*) \\
&= i d_A^* \psi.
\end{aligned}$$

□

We can further decompose the operators $L_K^{(1)}$ and $L_K^{(2)}$ into $(0,1)$ and $(1,0)$ parts to obtain the operators

$$\begin{aligned}
L_K^{(1)''} &= \bar\partial_A + \phi^*, \quad L_K^{(1)'} = \partial_A - \phi \\
L_K^{(2)''} &= \bar\partial_A - \phi, \quad L_K^{(2)'} = -\partial_A - \phi.
\end{aligned} \tag{3.10}$$

Note that all of the operators (3.10) satisfy the product rule, i.e. for $g \in \Omega^0(\mathrm{ad} P^{\mathbb{C}})$ we have

$$Lg = (Lg) + gL$$

and for $\omega \in \Omega^1(\mathrm{ad} P^{\mathbb{C}})$ we have

$$L\omega = (L\omega) - \omega L$$

where L denotes any of the operators in (3.10).

The proof of the next lemma is just an explicit calculation.

Lemma 3.9. *Let $d_A = \bar\partial_A + \partial_A$ and $\psi = i(\phi + \phi^*)$. We have*

$$u(d_A, \psi) = \frac{1}{2} i\Lambda \left(L_K^{(1)''} L_K^{(1)'} + L_K^{(1)'} L_K^{(1)''} + L_K^{(2)''} L_K^{(2)'} + L_K^{(2)'} L_K^{(2)''} \right).$$

The point of the decomposition (3.10) is to simplify the analysis of the action of the complex gauge group.

Lemma 3.10. *The complex gauge group acts on the operators* (3.10) *by*

$$\begin{aligned}
g \cdot L_K^{(1)''} &= g L_K^{(1)''} g^{-1}, \quad g \cdot L_K^{(1)'} = (g^*)^{-1} L_K^{(1)'} g^* \\
g \cdot L_K^{(2)''} &= (g^*)^{-1} L_K^{(2)''} g^*, \quad g \cdot L_K^{(2)'} = g L_K^{(2)'} g^{-1}.
\end{aligned}$$

Proof. Define the flat connection $D = d_A - i\psi = \bar{\partial}_A + \partial_A + \phi + \phi^*$ on $\mathrm{ad}P^{\mathbb{C}}$. Recall that the complex gauge group action is

$$g \cdot D = gDg^{-1}.$$

The induced action on d_A and $-i\psi$ is given by taking the skew-adjoint and self-adjoint parts respectively. Computing this via a direct calculation is quite complicated; it is simpler to write $D = L_K^{(1)''} - L_K^{(2)'}$ and take the $(0,1)$ and $(1,0)$ parts of the above equation. This gives us

$$g \cdot \left(L_K^{(1)''} - L_K^{(2)'} \right) = g L_K^{(1)''} g^{-1} - g L_K^{(2)'} g^{-1}$$

and so we have the desired action on $L_K^{(1)''}$ and $L_K^{(2)'}$.

The $(1,0)$ part of the connection form $-A' - \phi$ maps to $A'' - \phi^*$ under the Hermitian transpose, and so the action of g on $L_K^{(2)''}$ is the Hermitian transpose of the action on $L_K^{(2)'}$, i.e.

$$g \cdot L_K^{(2)''} = (g^*)^{-1} L_K^{(2)''} g^*.$$

The same analysis applied to the Hermitian transpose of $A'' + \phi^*$ leads to

$$g \cdot L_K^{(1)'} = (g^*)^{-1} L_K^{(1)'} g^*.$$

□

As a consequence of the previous lemmas, we have

Lemma 3.11. *Let $g \in \mathcal{G}^{\mathbb{C}}$. Then*

$$g^{-1} u(g \cdot d_A, g \cdot \psi) g = u(d_A, \psi) + \frac{1}{2} i \Lambda L_K^{(1)''} (h^{-1}(L_K^{(1)'} h)) + \frac{1}{2} i \Lambda L_K^{(2)'} (h^{-1}(L_K^{(2)''} h)) \tag{3.11}$$

The proof follows the same method as the proof of the well known formula

$$g^{-1} F_{g \cdot A} g = F_A + \bar{\partial}_A (h^{-1}(\partial_A h)), \tag{3.12}$$

where the operators $L_K^{(1)''}$ and $L_K^{(2)'}$ play the role of $\bar{\partial}_A$ and the operators $L_K^{(1)'}$ and $L_K^{(2)''}$ play the role of ∂_A. For the proof of (3.11) we only need the result of Lemma 3.10 and to check that these operators all satisfy the product rule, and then the method of proof of (3.12) will work for (3.11) also. The extra factor of $\frac{1}{2} i$ is due to the choice of constant in the definition of u (i.e. we could remove it by defining $u(d_A, \psi) = 2 d_A^* \psi$ instead).

Therefore, the evolution equation (3.8) for the change of metric h becomes

$$\frac{\partial h}{\partial t} = -h \left(u(d_{A_0}, \psi_0) + \frac{1}{2} i \Lambda L_K^{(1)''} (h^{-1}(L_K^{(1)'} h)) + \frac{1}{2} i \Lambda L_K^{(2)'} (h^{-1}(L_K^{(2)''} h)) \right), \tag{3.13}$$

where (A_0, ψ_0) is the initial condition for the flow and the operators $L_K^{(1)''}$, etc. are defined at (A_0, ψ_0).

The following version of the Kähler identities for the operators $L_K^{(1)''}$, etc. will be useful in simplifying (3.13) in analogy with [D1].

Lemma 3.12.

$$L_K^{(1)'^*} = i[\Lambda, L_K^{(1)''}], \quad L_K^{(1)''^*} = -i[\Lambda, L_K^{(1)'}]$$
$$L_K^{(2)'^*} = -i[\Lambda, L_K^{(2)''}], \quad L_K^{(2)''^*} = i[\Lambda, L_K^{(2)'}]$$

Proof. Firstly recall the well-known Kähler identities

$$\partial_A^* = i[\Lambda, \bar{\partial}_A], \quad \bar{\partial}_A^* = -i[\Lambda, \partial_A].$$

Define the operators $E'_\phi(\eta) = \phi \wedge \eta$ and $E''_\phi(\eta) = \phi^* \wedge \eta$ on the space of forms with values in the bundle. In [S2], Simpson proves identities for these operators that are analogous to the Kähler identities

$$E''^*_\phi = i[\Lambda, E'_\phi], \quad E'^*_\phi = -i[\Lambda, E''_\phi].$$

Combining these identities gives us

$$L_K^{(1)'^*} = (\partial_A - E'_\phi)^* = i[\Lambda, \bar{\partial}_A] + i[\Lambda, E''_\phi] = i[\Lambda, L_K^{(1)''}].$$

The proof of the rest of the identities in the lemma follows the same method. □

In particular, we have the following

Corollary 3.13. *Let h be a 0-form with values in a bundle associated to P. Then*

$$i\Lambda\left(L_K^{(1)''}L_K^{(1)'} + L_K^{(1)'}L_K^{(1)''}\right)h = -L_K^{(1)^*}L_K^{(1)}h + 2i\Lambda L_K^{(1)''}L_K^{(1)'}h$$
$$i\Lambda\left(L_K^{(2)''}L_K^{(2)'} + L_K^{(2)'}L_K^{(2)''}\right)h = -L_K^{(2)^*}L_K^{(2)}h + 2i\Lambda L_K^{(2)'}L_K^{(2)''}h.$$

Proof. The proof is just an explicit calculation using the Kähler identities. For $L_K^{(1)}$, the calculation is as follows.

$$\begin{aligned} i\Lambda\left(L_K^{(1)''}L_K^{(1)'} + L_K^{(1)'}L_K^{(1)''}\right)h &= L_K^{(1)'^*}L_K^{(1)'}h - L_K^{(1)''^*}L_K^{(1)''}h \\ &= 2L_K^{(1)'^*}L_K^{(1)'}h - L_K^{(1)^*}L_K^{(1)}h \\ &= 2i\Lambda L_K^{(1)''}L_K^{(1)'}h - L_K^{(1)^*}L_K^{(1)}h. \end{aligned}$$

For $L_K^{(2)}$, an analogous method gives

$$\begin{aligned} i\Lambda\left(L_K^{(2)''}L_K^{(2)'} + L_K^{(2)'}L_K^{(2)''}\right)h &= -L_K^{(2)'^*}L_K^{(2)'}h + L_K^{(2)''^*}L_K^{(2)''}h \\ &= -L_K^{(2)^*}L_K^{(2)}h + 2L_K^{(2)''^*}L_K^{(2)''}h \\ &= -L_K^{(2)^*}L_K^{(2)}h + 2i\Lambda L_K^{(2)'}L_K^{(2)''}h. \end{aligned}$$

□

This leads to the following simplification of (3.13).

Proposition 3.14.

$$\frac{\partial h}{\partial t} = -\frac{1}{2}hu(d_{A_0},\psi_0) - \frac{1}{2}u(d_{A_0},\psi_0)h - \frac{1}{4}\left(L_K^{(1)^*}L_K^{(1)} + L_K^{(2)^*}L_K^{(2)}\right)h + \frac{1}{2}i\Lambda(L_K^{(1)''}h)h^{-1}(L_K^{(1)'}h) + \frac{1}{2}i\Lambda(L_K^{(2)'}h)h^{-1}(L_K^{(2)''}h). \tag{3.14}$$

Proof. First note that the first term in (3.13) has the form

$$hu(d_{A_0},\psi_0) = \frac{1}{2}hu(d_{A_0},\psi_0) + \frac{1}{2}u(d_{A_0},\psi_0)h - \frac{1}{2}\left[u(d_{A_0},\psi_0),h\right].$$

Subsituting the results of Lemma 3.8 and Corollary 3.13 leads to

$$\begin{aligned} hu(d_{A_0},\psi_0) &= \frac{1}{2}hu(d_{A_0},\psi_0) + \frac{1}{2}u(d_{A_0},\psi_0)h - \frac{1}{4}i\Lambda\left(L_K^{(1)}L_K^{(1)} + L_K^{(2)}L_K^{(2)}\right)h \\ &= \frac{1}{2}hu(d_{A_0},\psi_0) + \frac{1}{2}u(d_{A_0},\psi_0)h + \frac{1}{4}\left(L_K^{(1)^*}L_K^{(1)}h + L_K^{(2)^*}L_K^{(2)}h\right) \\ &\quad - \frac{1}{2}i\Lambda L_K^{(1)''}L_K^{(1)'}h - \frac{1}{2}i\Lambda L_K^{(2)'}L_K^{(2)''}h \end{aligned}$$

We also have the following product rule expansions for the second and third terms

$$\begin{aligned} h\frac{1}{2}i\Lambda L_K^{(1)''}(h^{-1}(L_K^{(1)'}h)) &= -\frac{1}{2}i\Lambda\left((L_K^{(1)''}h)h^{-1}(L_K^{(1)'}h)\right) + \frac{1}{2}i\Lambda\left(L_K^{(1)''}L_K^{(1)'}h\right) \\ h\frac{1}{2}i\Lambda L_K^{(2)'}(h^{-1}(L_K^{(2)''}h)) &= -\frac{1}{2}i\Lambda\left((L_K^{(2)'}h)h^{-1}(L_K^{(2)''}h)\right) + \frac{1}{2}i\Lambda\left(L_K^{(2)'}L_K^{(2)''}h\right). \end{aligned}$$

Substituting all of these identities into (3.13) gives the required result.

□

Lemma 3.15. *Let $\psi = i(\phi+\phi^*)$ and define $E_\psi(\eta) = \psi\wedge\eta$ for $\eta\in\Omega^0(P)$. Also denote by E_ψ the induced operator on bundles associated to E. Then, on zero forms, we have*

$$L_K^{(1)^*}L_K^{(1)} + L_K^{(2)^*}L_K^{(2)} = 2d_A^*d_A + 2E_\psi^*E_\psi. \tag{3.15}$$

Proof. Note that $E_\psi = iE'_\phi + iE''_\phi$ and $E_\psi^*E_\psi = {E'_\phi}^*E'_\phi + {E''_\phi}^*E''_\phi$. A direct calculation shows that

$$\begin{aligned} L_K^{(1)^*}L_K^{(1)} &= \left(\bar\partial_A^* + \partial_A^* + {E''_\phi}^* - {E'_\phi}^*\right)\left(\bar\partial_A + \partial_A + E''_\phi - E'_\phi\right) \\ &= \bar\partial_A^*\bar\partial_A + \partial_A^*\partial_A + {E''_\phi}^*E''_\phi + {E'_\phi}^*E'_\phi \\ &\quad + \bar\partial_A^*E''_\phi - \partial_A^*E'_\phi + {E''_\phi}^*\bar\partial_A - {E'_\phi}^*\partial_A \end{aligned}$$

and

$$\begin{aligned} L_K^{(2)^*}L_K^{(2)} &= \left(\bar\partial_A^* - \partial_A - {E''_\phi}^* - {E'_\phi}^*\right)\left(\bar\partial_A - \partial_A - E''_\phi - E'_\phi\right) \\ &= \bar\partial_A^*\bar\partial_A + \partial_A^*\partial_A + {E''_\phi}^*E''_\phi + {E'_\phi}^*E'_\phi \\ &\quad - \bar\partial_A^*E''_\phi + \partial_A^*E'_\phi - {E''_\phi}^*\bar\partial_A + {E'_\phi}^*\partial_A. \end{aligned}$$

Adding these two results gives

$$\begin{aligned} {L_K^{(1)}}^* L_K^{(1)} + {L_K^{(2)}}^* L_K^{(2)} &= 2\bar{\partial}_A^* \bar{\partial}_A + 2\partial_A^* \partial_A + 2{E''_\phi}^* E''_\phi + 2{E'_\phi}^* E'_\phi \\ &= 2d_A^* d_A + 2E_\psi^* E_\psi. \end{aligned}$$

□

Therefore, the flow equation (3.14) for the metric can be written

$$\begin{aligned} \frac{\partial h}{\partial t} = &-\frac{1}{2} hu(d_{A_0}, \psi_0) - \frac{1}{2} u(d_{A_0}, \psi_0) h - \frac{1}{2}\Delta_{A_0} h - \frac{1}{2} E_{\psi_0}^* E_{\psi_0} h \\ &+ \frac{1}{2} i\Lambda(L_K^{(1)''} h) h^{-1} (L_K^{(1)'} h) + \frac{1}{2} i\Lambda(L_K^{(2)'} h) h^{-1} (L_K^{(2)''} h). \end{aligned} \tag{3.16}$$

Finally, we are ready to complete the proof of the second statement of Proposition 3.6. Since the right-hand side of (3.8) is τ-invariant, then $\frac{\partial h}{\partial t}$ is τ-invariant, and combining this with the fact that Δ_{A_0} is τ-invariant gives us

$$\frac{\partial h}{\partial t} = -\frac{1}{2}\Delta_{A_0} h + f(d_{A_0}, \psi_0, h), \tag{3.17}$$

where f is τ-invariant. Therefore, the metric evolution equation (3.17) descends to a non-linear parabolic equation on the non-orientable manifold, and so we have short-time existence for the solutions (see for example [Ha] or [T]). The same method as in [D1] or [Hong] shows that the change of metric then lifts to a solution to the flow equations. This completes the proof of the second statement of Proposition 3.6.

4. Real locus of an involution on the moduli space

In this section we restrict to the case where $\tilde{\Sigma}$ is a compact surface and describe the relation between the Betti moduli space over a nonorientable surface Σ and the fixed point set of the Betti moduli space over its orientable double cover $\tilde{\Sigma}$ (with respect to the involution τ induced by the deck transformation on the double cover). Also, we would like to give geometric meaning for the components of the fixed point set of the moduli space.

The proofs for surfaces Σ_2^ℓ are similar to those for Σ_1^ℓ. In order to avoid repetition of similar proofs, here we only give the proofs for surfaces Σ_1^ℓ and omit the proofs for surfaces Σ_2^ℓ.

Since G is complex and semi-simple, then G has finite center ([Kn, Prop 7.5]). Recall that we defined a natural map in Section 2

$$I: \ \mathrm{Hom}(\pi_1(\Sigma_i^\ell), G) /\!\!/ G \to (\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G) /\!\!/ G)^\tau.$$

and a homeomorphism (Proposition 2.11) between $\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)/\!\!/G$ (the Betti moduli space) and the the two based point moduli spaces $Z_{\mathrm{flat}}^{\ell,1}(G)$, $Z_{\mathrm{flat}}^{\ell,2}(G)$ with respect to the $G \times G$ action.

Let $x = (a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \in Z^{\ell,1}_{\text{flat}}(G)$. It is natural to call x simple if the corresponding point

$$\Phi^{\ell,1}_G(x) = (a_1, b_1, \cdots, a_\ell, b_\ell, cb'_\ell c^{-1}, ca'_\ell c^{-1}, \cdots, cb'_1 c^{-1}, ca'_1 c^{-1})$$

in $\operatorname{Hom}(\pi_1(\tilde{\Sigma}), G)$ is a simple point(i.e. of minimal isotropy $Z(G)$). In other words, x should satisfy

$$\begin{aligned} &\bigcap_{i=1}^{\ell}(G_{a_i} \cap G_{b_i} \cap G_{ca'_i c^{-1}} \cap G_{cb'_i c^{-1}}) = Z(G), \quad \text{and} \\ &\bigcap_{i=1}^{\ell}(G_{a'_i} \cap G_{b'_i} \cap G_{c' a_i c'^{-1}} \cap G_{c' b_i c'^{-1}}) = Z(G). \end{aligned} \tag{4.1}$$

Similarly, it is natural to call a point in $\operatorname{Hom}(\pi_1(\Sigma^\ell_1), G)$ simple if its lifting in $\operatorname{Hom}(\pi_1(\tilde{\Sigma}), G)$ (cf. Section 2) is simple. In other words, x should satisfy

$$\bigcap_{i=1}^{\ell}(G_{a_i} \cap G_{b_i} \cap G_{ca_i c^{-1}} \cap G_{cb_i c^{-1}}) = Z(G). \tag{4.2}$$

In the companion paper [HW] we show that these are precisely the smooth points in the corresponding moduli spaces $\operatorname{Hom}(\pi_1(\Sigma^\ell_i), G)/\!\!/G$.

Definition 4.1. *A point in* $Z^{\ell,1}_{\text{flat}}(G)$ *is called* simple *if its stabilizer with respect to the* $G \times G$*-action equals to* $Z(G) \times Z(G)$ *(i.e. satisfies equation (4.1)). Let* $Z^{\ell,1}_{\text{flat}}(G)^{\text{simp}}$ *denote the subset of* $Z^{\ell,1}_{\text{flat}}(G)$ *that contains all points that are simple, and* $Z^{\ell,1}_{\text{flat}}(G)^{\text{smooth}}$ *denote the subset that contains all points that are reductive and simple. Notice that points in* $\operatorname{Hom}(\pi_1(\tilde{\Sigma}), G)$ *that are reductive and simple in the usual sense are precisely the smooth points in* $\operatorname{Hom}(\pi_1(\tilde{\Sigma}), G)/\!\!/G$*.*

Definition 4.2. *A point in* $\operatorname{Hom}(\pi_1(\Sigma^\ell_1), G)$ *is called* simple *if it satisfies equation (4.2). Let* $\operatorname{Hom}(\pi_1(\Sigma^\ell_1), G)^{\text{simp}}$ *denote the subset of* $\operatorname{Hom}(\pi_1(\Sigma^\ell_1), G)$ *that contains all points that are simple, and* $\operatorname{Hom}(\pi_1(\Sigma^\ell_1), G)^{\text{smooth}}$ *denote the subset that contains all points that are reductive(or has closed orbits) and simple.*

Recall the notation $(V, c, V', c') \overset{\text{def}}{=} (a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \in Z^{\ell,1}_{\text{flat}}(G)$. If $(V, c, V', c') \in (Z^{\ell,1}_{\text{flat}}(G)/\!\!/G \times G)^\tau$, then there exists (g_1, g_2) such that $(g_1, g_2) \cdot (V, c, V', c') = (V'c', V, c)$, i.e.

$$g_1 g_2 V' g_2^{-1} g_1^{-1} = V', \quad g_1 g_2 c' g_1^{-1} g_2^{-1} = c', \quad g_2 g_1 V g_1^{-1} g_2^{-1} = V, \quad g_2 g_1 c g_2^{-1} g_1^{-1} = c$$

This implies that

$$g_1 g_2 \in G_{a'_i} \cap G_{b'_i} \cap G_{c' a_i c'^{-1}} \cap G_{c' b_i c'^{-1}} \quad \text{and} \quad g_2 g_1 \in G_{a_i} \cap G_{b_i} \cap G_{ca'_i c^{-1}} \cap G_{cb'_i c^{-1}}.$$

Thus, if $(V, c, V', c') \in Z^{\ell,1}_{\text{flat}}(G)^{\text{simp}}$ then $g_1 g_2 \in Z(G)$ and $g_2 g_1 \in Z(G)$. Thus, $(\text{id}, g_1^{-1}) \cdot (V, c, V', c') = (V, cg_1, g_1^{-1} V' g_1, g_1^{-1} c') = (V, cg_1, V, cg_1(g_1^{-1} g_2^{-1}))$. In other

words, if $(V,c,V',c') \in (Z^{\ell,1}_{\text{flat}}(G)^{\text{simp}}/\!\!/G\times G)^\tau$, then (V,c,V',c') is equivalent to (V,d,V,dr) for some $d\in G$ and $r\in Z(G)$. This tells us that:

$$(Z^{\ell,1}_{\text{flat}}(G)^{\text{simp}}/\!\!/G\times G)^\tau = (Z^{\ell,1}_{\text{flat}}(G)^{\text{smooth}}/G\times G)^\tau = Pr(\bigcup_{r\in Z(G)} N_r^{\text{smooth}})$$

where $Pr : Z^{\ell,1}_{\text{flat}}(G)^{\text{red}} \to Z^{\ell,1}_{\text{flat}}(G)^{\text{red}}/G\times G = Z^{\ell,1}_{\text{flat}}(G)/\!\!/G\times G$ is the projection to the quotient space, and

$$N_r = \{(V,c,V,cr)\}\cap Z^{\ell,1}_{\text{flat}}(G), \quad N_r^{\text{smooth}} = N_r \cap Z^{\ell,1}_{\text{flat}}(G)^{\text{smooth}}.$$

Lemma 4.3. *Let $N_r^{\text{red}} = N_r\cap Z^{\ell,1}_{\text{flat}}(G)^{\text{red}}$. Then $Pr(N_r^{\text{red}}) = Pr(N_s^{\text{red}})$ iff $r=st^2$ for some $t\in Z(G)$, and $Pr(N_r^{\text{smooth}})\cap Pr(N_s^{\text{smooth}}) = \emptyset$ iff $r\neq st^2$ for some $t\in Z(G)$.*

Proof. We first show that if $r=st^2$ for some $t\in Z(G)$, then $Pr(N_r^{\text{red}}) = Pr(N_s^{\text{red}})$. If $(V,c,V,cr)\in N_r$, then $(V,ct,V,cts)\in N_S$ and

$$(\text{id},t)\cdot(V,ct,V,cts) = (V,ctt^{-1},V,tcts) = (V,c,V,ct^2s) = (V,c,V,cr),$$

i.e. $Pr(N_r^{\text{red}})\subset Pr(N_s^{\text{red}})$. Similarly, $Pr(N_s^{\text{red}})\subset Pr(N_r^{\text{red}})$. Thus, $Pr(N_r^{\text{red}}) = Pr(N_s^{\text{red}})$.

Next we want to show that $Pr(N_r^{\text{smooth}})\cap Pr(N_s^{\text{smooth}}) = \emptyset$ iff $r\neq st^2$ for some $t\in Z(G)$. Notice that the condition (4.1) for simple points now becomes the condition (4.2): $\cap_{i=1}^\ell(G_{a_i}\cap G_{b_i}\cap G_{ca_ic^{-1}}\cap G_{cb_ic^{-1}} = Z(G))$. Suppose $[x]\in Pr(N_r^{\text{smooth}})\cap Pr(N_s^{\text{smooth}})$. Then there are $(V,c,V,cr)\in N_r^{\text{smooth}}$ and $(V'c',V',c's)\in N_s^{\text{smooth}}$ such that $x=(g_1,g_2)\cdot(V,c,V,cr) = (V',c',V',c's)$. This gives us $g_1^{-1}g_2Vg_2^{-1}g_1 = V$ and $g_1^{-1}g_2c = c(g_1^{-1}g_2)^{-1}sr^{-1}$. The first equality implies that $g_1^{-1}g_2\in\cap_{i=1}^\ell(G_{a_i}\cap G_{b_i})$. The second equality plus the first equality implies that $g_1^{-1}g_2\in\cap_{i=1}^\ell(G_{cb_ic^{-1}}\cap G_{ca_ic^{-1}})$. Thus, $g_1^{-1}g_2\in\cap_{i=1}^\ell(G_{a_i}\cap G_{b_i}\cap G_{ca_ic^{-1}}\cap G_{cb_ic^{-1}})$, and since this point is simple, it implies that $g_1^{-1}g_2\in Z(G)$. However, $g_1^{-1}g_2cg_1^{-1}g_2 = csr^{-1}$. we conclude that $r = s(g_1^{-1}g_2)^{-2}$, a contradiction. ☐

We conclude that

$$(Z^{\ell,1}_{\text{flat}}(G)^{\text{smooth}}/G\times G)^\tau = \bigcup_{r\in\frac{Z(G)}{2Z(G)}} Pr(N_r^{\text{smooth}})$$

is a disjoint union. By the definition of the map I, $I(\text{Hom}(\pi_1(\Sigma_1^\ell),G)^{\text{simp}}/\!\!/G) = Pr(N_{e_G}^{\text{smooth}})$. Thus, we see that the map I is surjective if and only if $Z(G)=2Z(G)$.

Let $[(V,c)]$ and $[(V',c')]$ be two points in $\text{Hom}(\pi_1(\Sigma_1^\ell),G)^{\text{simp}}/\!\!/G$. If they map to the same point under I, then there exists $(g_1,g_2)\in G\times G$ such that $(g_1,g_2)\cdot(V,c,V,c) = (V',c',V',c')$. Again we have $g_1^{-1}g_2V(g_1^{-1}g_2)^{-1} = V$ and $g_1^{-1}g_2c = c(g_1^{-1}g_2)^{-1}$. Similar to the argument in Lemma 4.3, we see that $g_1^{-1}g_2\in Z(G)$ and $(g_1^{-1}g_2)^2 = e_G$. If (V,c) and (V',c') are equivalent under the G action, then it's necessary that $g_1=g_2$, otherwise it will contradict to the assumption that

they are both simple. Thus, for each point in $Pr(N_{e_G}^{\text{smooth}})$, the cardinality of its preimage in $\text{Hom}(\pi_1(\Sigma_1^\ell), G)^{\text{simp}} /\!\!/ G$ under I is precisely the number of elements in the center of G for which its square is the identity element. In other words, the map I is a $|Z(G)/2Z(G)| : 1$ covering map from $\text{Hom}(\pi_1(\Sigma), G)^{\text{smooth}}/G$ onto $Pr(N_{e_G}^{\text{smooth}}) \subset (\text{Hom}(\pi_1(\tilde{\Sigma}), G)^{\text{smooth}}/G)^\tau$.

The fixed point set is a disjoint union $\bigcup_{r \in \frac{Z(G)}{2Z(G}} Pr(N_r^{\text{smooth}})$, but we have only identified $Pr(N_{e_G}^{\text{smooth}})$ with the image of the moduli space $\text{Hom}(\pi_1(\Sigma), G)^{\text{smooth}}/G$ over the nonorientable surface. In the rest of this section, we would like to give a description for the full fixed point set

$$\left(\text{Hom}(\pi_1(\tilde{\Sigma}), G)^{\text{smooth}}/G\right)^\tau = \bigcup_{r \in \frac{Z(G)}{2Z(G)}} Pr(N_r^{\text{smooth}}).$$

Consider now a random complex semi-simple Lie group H and denote its universal covering by $\tilde{H}$. Let $\rho : \tilde{H} \to H$ be the universal covering map and we also use ρ to denote the map from $\tilde{H}^{2\ell+1}$ to $H^{2\ell+1}$. Consider

$$C_r \stackrel{\text{def}}{=} \{(a_1, \cdots, b_\ell, c) \in \tilde{H}^{2\ell+1} | \prod_{i=1}^{\ell} [a_i, b_i] = c^2 r\} \subset \tilde{H}^{2\ell+1}$$

for each $r \in \pi_1(H) \cong \text{Ker}\rho \subset Z(\tilde{H})$.

Lemma 4.4. *Let $D_r \stackrel{\text{def}}{=} \rho(C_r) \subset \text{Hom}(\pi_1(\Sigma_1^\ell), H)$. Then,*

(1) *If $r_1 \neq r_2$ in $\pi_1(H)/2\pi_1(H)$ then $D_{r_1} \cap D_{r_2} = \emptyset$.*
(2)
$$\text{Hom}(\pi_1(\Sigma_1^\ell), H) /\!\!/ H = \bigcup_{r \in \frac{\pi_1(H)}{2\pi_1(H)}} D_r /\!\!/ H$$
is a disjoint union.

Proof. It is clear that $\rho^{-1}(\text{Hom}(\pi_1(\Sigma_1^\ell), H)) = \bigcup_{r \in \text{Ker}\rho} C_r$, so (2) is true if (1) is proved. Now to prove (1), assume that it is false, i.e. $D_{r_1} \cap D_{r_2} \neq \emptyset$, then there exist $(a_1, \cdots, b_\ell, c) \in C_{r_1}$ and $(a_1', \cdots, b_\ell', c') \in C_{r_2}$ such that $\rho((a_1, \cdots, b_\ell, c)) = \rho((a_1', \cdots, b_\ell', c'))$. In other words, $a_i = a_i' s_i$, $b_i = b_i' t_i$, $c = c'u$ for some s_i, t_i, $u \in \text{Ker}\rho \subset Z(\tilde{H})$. Thus, $r_1 = \prod_{i=1}^{\ell} [a_i, b_i] c^{-2} = \prod_{i=1}^{\ell} [a_i' s_i, b_i' t_i](c'u)^{-2} = \prod_{i=1}^{\ell} [a_i', b_i'](c')^{-2} u^{-2} = r_2 u^{-2}$, which is a contradiction. □

Definition 4.5. *A point x in C_r is called simple if it descends to a simple point in $\rho(C_r) = D_r$. In other words, if $x = (a_1, b_1, \cdots, a_\ell, b_\ell, c) \in C_r$ is called simple if $\rho(x) \in D_r$ is simple, i.e. if $\cap_{i=1}^{\ell} (H_{\rho(a_i)} \cap H_{\rho(b_i)} \cap H_{\rho(c a_i c^{-1})} \cap H_{\rho(c b_i c^{-1})}) = Z(H)$.*

Now back to the case we are interested in. If apart from being complex semi-simple, our structure group G for the moduli space of Higgs bundles is also simply

connected, then we can consider the moduli space whose structure group is the projective group $PG = G/Z(G)$. Let $H = PG$, $\tilde{H} = G$, then $\pi_1(H) = Z(G)$ and

$$\mathrm{Hom}(\pi_1(\Sigma_1^\ell), PG)/\!\!/PG = \bigcup_{r\in\frac{Z(G)}{2Z(G)}} D_r/\!\!/PG$$

is a disjoint union following Lemma 4.4.

In general, there is no relation between the intersections $\cap_{i=1}^\ell (H_{\rho(a_i)} \cap H_{\rho(b_i)} \cap H_{\rho(ca_ic^{-1})} \cap H_{\rho(cb_ic^{-1})})$ and $\cap_{i=1}^\ell(\tilde{H}_{a_i} \cap \tilde{H}_{b_i} \cap \tilde{H}_{ca_ic^{-1}} \cap \tilde{H}_{cb_ic^{-1}})$. However, in the special case that $H = PG$ and $\tilde{H} = G$, there is some relation:

Lemma 4.6. *Assume that $H = G/Z(G)$ where G is a complex, semi-simple, and simply connected Lie group. If $x = (a_1, b_1, \cdots, a_\ell, b_\ell, c) \in C_r$ is a simple point, then $\cap_{i=1}^\ell(G_{a_i} \cap G_{b_i} \cap G_{ca_ic^{-1}} \cap G_{cb_ic^{-1}}) = Z(G)$.*

Proof. Let $g \in \cap_{i=1}^\ell(G_{a_i} \cap G_{b_i} \cap G_{ca_ic^{-1}} \cap G_{cb_ic^{-1}})$. Thus, $ga_ig^{-1} = a_i$, $gb_ig^{-1} = b_i$, $gcb_ic^{-1}g^{-1} = cb_ic^{-1}$, $gca_ic^{-1}g^{-1} = ca_ic^{-1}$. Thus, $\rho(ga_ig^{-1}) = \rho(a_i)$, $\rho(gb_ig^{-1}) = \rho(b_i)$, $\rho(gcb_ic^{-1}g^{-1}) = \rho(cb_ic^{-1})$, $\rho(gca_ic^{-1}g^{-1}) = \rho(ca_ic^{-1})$. Thus $\rho(g) \in Z(H)$ since x is simple. However, since $H = G/Z(G)$, we have $Z(H)$ is the trivial group and so $g \in \mathrm{Ker}\rho \cong \pi_1(H) \cong Z(G)$. Thus $Z(G) \subset \cap_{i=1}^\ell(G_{a_i} \cap G_{b_i} \cap G_{ca_ic^{-1}} \cap G_{cb_ic^{-1}}) \subset Z(G)$. $\square$

Remark 4.7. *Notice that, the converse is false. The condition that $\cap_{i=1}^\ell(G_{a_i} \cap G_{b_i} \cap G_{ca_ic^{-1}} \cap G_{cb_ic^{-1}}) = Z(G)$ does not imply that $(a_1, b_1, \cdots, a_\ell, b_\ell, c)$ is simple.*

Recall that

$$\begin{aligned} P_r(N_r^{\mathrm{smooth}}) &= \{[(a_1, \cdots, b_\ell, c, a_1, \cdots, b_\ell, cr)]\} \subset Z_{\mathrm{flat}}^{\ell,1}(G)^{\mathrm{smooth}}/G \times G \\ &\xrightarrow{\cong} \{[(a_1, \cdots, b_\ell, cb_\ell c^{-1}, \cdots, ca_1c^{-1})]\} \subset \mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\mathrm{simp}}/\!\!/G \\ &= \{(a_1, \cdots, b_\ell, cb_\ell c^{-1}, \cdots, ca_1c^{-1}) \in (G)^{4\ell} | \prod_{i=1}^{\ell}[a_i, b_i] = c^2 r\}^{\mathrm{simp}}/\!\!/G \qquad (4.3) \end{aligned}$$

Consider the surjective map

$$\Psi:\ C_r/\!\!/G \to \{(a_1, \cdots, b_\ell, cb_\ell c^{-1}, \cdots, ca_1c^{-1}) \in (G)^{4\ell} | \prod_{i=1}^{\ell}[a_i, b_i] = c^2 r\}/\!\!/G$$

$$[(a_1, \cdots, b_\ell, c)] \mapsto [(a_1, \cdots, b_\ell, cb_\ell c^{-1}, \cdots, ca_1c^{-1})],$$

It is not one to one. In fact, we have the following:

Lemma 4.8. *The map Ψ is a $|Z(G)/2Z(G)| : 1$ covering map on the simple points.*

Proof. Let $[(V, c)]$ and $[(V', c')]$ be two simple points in $C_r/\!\!/G$. Suppose that their images under this map Ψ are the same $[(V, c\mathfrak{r}(V)c^{-1})] = [(V', c'\mathfrak{r}(V')c'^{-1})]$. Then

there exists $g \in G$ such that $g(V, c\mathfrak{r}(V)c^{-1})g^{-1} = (V', c'\mathfrak{r}(V')c'^{-1})$. Thus we have $(c')^{-1}gcg^{-1} \in G_{b'} \cap G_{a'}$. On the other hand, $(V, c),\ (V'c') \in\ C_r$ implies that

$$\mathfrak{m}(V)c^{-2} = r = \mathfrak{m}(V')(c')^{-2} = g\mathfrak{m}(V)g^{-1}(c')^{-2} = gc^2rg^{-1}(c')^{-2}$$

and thus $c'gc^{-1}g^{-1} = (c')^{-1}gcg^{-1}$.

We also have $c'gc^{-1}g^{-1} \in G_{c'b'_i(c')^{-1}}$ since $(c'gc^{-1}g^{-1})c'b'_i(c')^{-1}(c'gc^{-1}g^{-1})^{-1} = c'b'_i(c')^{-1}$, and similarly $c'gc^{-1}g^{-1} \in G_{c'a'_i(c')^{-1}}$.

Therefore we see that $c'gc^{-1}g^{-1} \in \cap_{i=1}^{\ell}(G_{a_i} \cap G_{b_i} \cap G_{ca_ic^{-1}} \cap G_{cb_ic^{-1}})$. Lemma 4.6 then tells us that $c'gc^{-1}g^{-1} \in Z(G)$. Denote $s = c'gc^{-1}g^{-1} \in Z(G)$, then $\mathfrak{m}(V') = (c')^2r = gc^2g^{-1}s^2r$ and $\mathfrak{m}(V') = g\mathfrak{m}(V)g^{-1} = gc^2rg^{-1}$. Thus, $s^2 = e_G$. In other words, we have $a'_i = ga_ig^{-1},\ b'_i = gb_ig^{-1},\ c' = gcg^{-1}s$ with $s \in Z(G)$ and $s^2 = e_G$. Again, if $s \neq e_G$ then $[(V, c)] \neq [(V', c')]$, i.e. each s represents one different equivalent class in $C_r^{\text{simp}}/\!\!/G$. Thus, this map is a $|Z(G)/2Z(G)| : 1$ covering map on the simple points. □

To summarize the discussion in this section, we conclude with the following theorem:

Theorem 4.9. *Let G be a complex semi-simple Lie group. The fixed point set $(\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\text{smooth}}/G)^{\tau}$ of the smooth moduli space $\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\text{smooth}}/G$ under the involution τ consists of complex submanifolds (since G is complex and we consider only the simple points) $\mathcal{N}_0$, $\mathcal{N}_{\lambda_r}$ $(1 \leq \lambda_r \leq |Z(G)/2Z(G)| - 1)$ where*

1. *$\mathcal{N}_0 \stackrel{\text{def}}{=} P_r(N_{e_G}^{\text{smooth}})$, $\mathcal{N}_{\lambda_r} \stackrel{\text{def}}{=} P_r(N_r^{\text{smooth}})$, and λ_r is the image of $r \in Z(G)$ under the map $Z(G)/2Z(G) \to |Z(G)/2Z(G)| \subset \mathbb{N} \cup \{0\}$ where the identity element e_G is sent to 0.*
2. *The smooth moduli space $\mathrm{Hom}(\pi_1(\Sigma), G)^{\text{simp}}/\!\!/G$ over the nonorientable surface is a $|Z(G)/2Z(G)| : 1$ covering space of the submanifold $\mathcal{N}_0$. In particular, if $|Z(G)|$ is odd, then $\mathrm{Hom}(\pi_1(\Sigma), G)^{\text{simp}}/\!\!/G$ is homeomorphic to the fixed point set $(\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\text{smooth}}/G)^{\tau}$ since it only contains $\mathcal{N}_0$ in this case.*
3. *Assume further that G is simply connected. Then the Betti moduli space $\mathrm{Hom}(\pi_1(\Sigma), PG)/\!\!/PG$ for the nonorientable surface Σ with structure group the projective group $PG = G/Z(G)$ is a disjoint union of $\rho(C_r/\!\!/G)$ for all $r \in Z(G)/2Z(G)$, where ρ is the universal covering map $G \to PG$. Moreover, each $C_r^{\text{simp}}/\!\!/G$ is a $|Z(G)/2Z(G)| : 1$ covering space of the submanifold $\mathcal{N}_{\lambda_r}$.*

Remark 4.10. (1) *Notice that $C_{e_G}/\!\!/G$ is exactly $\mathrm{Hom}(\pi_1(\Sigma), G)/\!\!/G$, so when G is simply connected, statement (3) above implies statement (2).*

(2) *Geometrically, $C_r /\!\!/ G$ can be viewed as the representation moduli space over a punctured surface, where the holonomy along the boundary is described by r.*

Since $\rho(r) = e_H$ for all $r \in Z(G) = \mathrm{Ker}\rho$ here, we have $\rho((V, c, V, cr)) = (\rho(V), \rho(c), \rho(V), \rho(c))$ and so $\rho(\bigcup_{r \in \frac{Z(G)}{2Z(G)}} P_r(N_r^{\mathrm{smooth}})) \subset P_r(N_{e_H}^{\mathrm{smooth}})$. On the other hand, let (V, c, V, c) be any element in $Pr(N_{e_H}^{\mathrm{smooth}})$ and pick any lifting $\tilde{a}_i$, $\tilde{b}_i$ and $\tilde{c}$ in G for a_i, b_i and c. $\mathfrak{m}(V)c^{-2} = e_H$ implies that $\mathfrak{m}(\tilde{V})\tilde{c}^{-2} = r$ for some $r \in \mathrm{Ker}\rho$, i.e. $(\tilde{V}, \tilde{c}, \tilde{V}, \tilde{c}r) \in N_r$ and so $P_r(N_{e_H}^{\mathrm{smooth}}) \subset \rho(\bigcup_{r \in \frac{Z(G)}{2Z(G)}} P_r(N_r^{\mathrm{smooth}}))$. Thus, we can reformulate Theorem 4.9 with the following diagram:

$$
\begin{array}{ccc}
C_{e_G}^{\mathrm{simp}} /\!\!/ G = \frac{\mathrm{Hom}(\pi_1(\Sigma), G)^{\mathrm{smooth}}}{G} & \xrightarrow{|\frac{Z(G)}{2Z(G)}|:1} & (\frac{\mathrm{Hom}(\pi_1(\tilde{\Sigma}), G)^{\mathrm{smooth}}}{G})^{\tau} \\
\cap & & \| \\
\bigcup_{r \in \frac{Z(G)}{2Z(G)}} C_r^{\mathrm{simp}} /\!\!/ G & \xrightarrow{|\frac{Z(G)}{2Z(G)}|:1} & \bigcup_{r \in \frac{Z(G)}{2Z(G)}} P_r(N_r^{\mathrm{smooth}}) \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
\frac{\mathrm{Hom}(\pi_1(\Sigma), H)^{\mathrm{smooth}}}{H} & \xrightarrow{\cong} & (\frac{\mathrm{Hom}(\pi_1(\tilde{\Sigma}), H)^{\mathrm{smooth}}}{H})^{\tau} = P_r(N_{e_H}^{\mathrm{smooth}}) \\
\| & & \| \\
\rho(\bigcup_{r \in \frac{Z(G)}{2Z(G)}} C_r^{\mathrm{simp}} /\!\!/ G) & & \rho(\bigcup_{r \in \frac{Z(G)}{2Z(G)}} P_r(N_r^{\mathrm{smooth}}))
\end{array}
$$

5. Symplectic Structure

In this section, we show that there is a natural symplectic structure on the Hitchin moduli space over a nonorientable surface. We will show this via two different approaches. The first approach is to show that the smooth part of the Hitchin moduli space over a nonorientable surface is a fixed point set of a symplectic involution of a symplectic manifold. The second approach is to realize the Hitchin moduli space over a nonorientable surface as a symplectic quotient.

Recall the three distinct symplectic structures on the cotangent bundle $T^*\mathcal{A}(\tilde{P})$ of the space of connections $\mathcal{A}(\tilde{P})$ over a Riemann surface $\tilde{\Sigma}$:

For (θ, ξ), $(\vartheta, \eta) \in T_{(A,\phi)}(T^*\mathcal{A}(\tilde{P})) \cong \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P}) \oplus \Omega^1(\tilde{\Sigma}, \mathrm{ad}\tilde{P})$, the three natural symplectic forms are defined as:

$$
\begin{aligned}
\omega_1((\theta, \xi), (\vartheta, \eta)) &= \int_{\tilde{\Sigma}} \mathrm{Tr}\ \theta \wedge \vartheta - \xi \wedge \eta \\
\omega_2((\theta, \xi), (\vartheta, \eta)) &= \int_{\tilde{\Sigma}} \mathrm{Tr}\ \theta \wedge \eta - \vartheta \wedge \xi \\
\omega_3((\theta, \xi), (\vartheta, \eta)) &= \int_{\tilde{\Sigma}} \mathrm{Tr}\ \theta \wedge \star\eta - \xi \wedge \star\vartheta
\end{aligned}
$$

with three corresponding moment maps that all make $T^*\mathcal{A}(\tilde{P})$ into a Hamiltonian $\tilde{\mathcal{G}}$-space:

$$\begin{aligned}\mu_1(A,\phi) &= F_A - \phi\wedge\phi\\ \mu_2(A,\phi) &= d_A\phi\\ \mu_3(A,\phi) &= d_A^*\phi\end{aligned}$$

The Hitchin moduli space $\mathcal{M}(\tilde{\Sigma},\tilde{P})$ over $\tilde{\Sigma}$ can be viewed as a hyperkähler quotient of $T^*\mathcal{A}(\tilde{P})$ by the gauge action though we do not use the compatible complex structures here. To be precise,

$$\mathcal{M}(\tilde{\Sigma},\tilde{P}) \cong \cap_{i=1}^3 \mu_i^{-1}(0)/\mathcal{G}(\tilde{P}).$$

The three symplectic forms ω_1, ω_2, ω_3 on $T^*\tilde{\mathcal{A}}(\tilde{P})$ descend to Ω_1, Ω_2, and Ω_3 on (the smooth part of) the moduli space $\mathcal{M}(\tilde{\Sigma},\tilde{P})$ via hyperkähler quotient theory [Fu, Hi1, HKLR]

5.1. **Symplectic involution.**

Lemma 5.1. *Let M be a smooth manifold and H be a compact Lie group acting on M. If m is a fixed point of the H action, then H induces an action on T_mM and $T_m(M^H) = (T_mM)^H$.*

Proof. Since H is compact, Bochner's Linearization Theorem tells us that, there exists a H-invariant open neighborhood U of m, and a H-equivariant diffeomorphism F from U onto an open neighborhood V of 0 in T_mM, such that $F(m) = 0$, $dF_m = id : T_mM \to T_mM$. Clearly $T_m(M^H) \subset (T_mM)^H$.

First we show that the restriction $F : U \cap M^H \to V \cap (T_mM)^H$ is also a diffeomorphism. If $m \in U \cap M^H$, i.e. $h.m = m$ then $F(m) = F(h.m) = h.F(m)$, i.e. $F(m)$ is fixed by H, so $F(U \cap M^H) \subset V \cap (T_mM)^H$. If $v \in V \cap (T_mM)^H$, then since $F : U \to V$ is a diffeomorphism, there exist a unique $m \in U$ such that $F(m) = v$. Thus $F(h.m) = h.F(m) = h.v = v = F(m)$ which implies $m \in U \cap M^H$. Thus $F : U \cap M^H \to V \cap (T_mM)^H$ is a diffeomorphism since U and V are open neighborhoods.

Let v be a vector in $V \cap (T_mM)^H$, since $V \cap (T_mM)^H \cong T_0(V \cap (T_mM)^H)$, there exists a curve $v(t) \in V \cap (T_mM)^H$ such that $v(0) = 0$ and $v'(0) = v$. Now since $F : U \cap M^H \to V \cap (T_mM)^H$ is a diffeomorphism, there exists a curve $r(t) \in U \cap M^H$ such that $F(r(t)) = v(t)$. Thus, $r(0) = m$ and $dF_m(r'(0)) = v$. Now since dF_m is the identity map, it implies that $r'(0) = v$ and thus $T_m(M^H) = (T_mM)^H$.

Notice that, if M is finite dimensional, the proof can be simplified as follows: since F is a H-equivariant diffeomorphism then $\dim M^H = \dim(T_mM)^H$, i.e. $\dim T_m(M^H) = \dim(T_mM)^H$ and so $T_m(M^H) = (T_mM)^H$ for dimensional reasons since $T_m(M^H) \subset (T_mM)^H$. □

Proposition 5.2. *Let (M^{2n}, ω) be a symplectic manifold and $\tau : M \to M$ be a symplectic isometric involution on M. If the fixed point set M^τ of τ is nonempty, then M^τ is even dimensional. In particular, $(M^\tau, \imath^*\omega)$ is a symplectic manifold (possibly disconnected), where $\imath : M^\tau \to M$ is the inclusion.*

Proof. Since τ is an isometric involution, at $m \in M^\tau$, we have a decomposition of $T_m M$ into eigenspaces:

$$T_m M = \{v \in T_m M \mid \tau_* v = -v\} \oplus \{v \in T_m M \mid \tau_* v = v\}$$

and we denote the eigenspaces by $(T_m M)^-$ and $(T_m M)^+$ respectively. Lemma 5.1 shows that $T_m(M^\tau) = (T_m M)^+$ since $\mathbb{Z}_2$ is compact.

It is straightforward to check that $\omega(v, u) = 0$ if $v \in (T_m M)^+$ and $u \in (T_m M)^-$: Let $v \in (T_m M)^+$ and $u \in (T_m M)^-$ then $\omega(v, u) = \tau^*\omega(v, u) = \omega(\tau_* v, \tau_* u) = \omega(v, -u) = -\omega(v, u)$ which implies that $\omega(v, u) = 0$.

Restricting the symplectic form ω on M^τ, we get a two form $\imath^*\omega$ which is closed, since $d\imath^*\omega = \imath^* d\omega = 0$. To say that $\imath^*\omega$ is non-degenerate on M^τ, we just need to show that, if $\omega_m(v, w) = 0$ for all $w \in T_m(M^\tau) = (T_m M)^+$, $m \in M^\tau$, then $v = 0$ or $v \in (T_m M)^-$. Suppose this is not true, then there exists a nonzero vector $v_0 \in (T_m M)^+$ such that $\omega_m(v_0, w) = 0$ for all $w \in (T_m M)^+$. Let u be any vector in $T_m M$, then $u = u^+ + u^-$ where $u^\pm \in (T_m M)^\pm$. We have $\omega_m(v_0, u) = \omega_m(v_0, u^+) + \omega_m(v_0, u^-)$ and $\omega_m(v_0, u^+) = 0$ by assumption. We have already shown that $\omega_m(x, y) = 0$ for any $x \in (T_m M)^+$ and $y \in (T_m M)^-$. Thus $\omega_m(v_0, u) = 0 + \omega_m(v_0, u^-) = 0$ for all $u \in T_m M$. However, ω is non-degenerate on M, a contradiction. Thus v_0 must be zero or $v_0 \in (T_m M)^-$. We conclude that $(M^\tau, \imath^*\omega)$ is symplectic and thus even dimensional if M^τ is nonempty. □

Lemma 5.3. *Let $\mathcal{M}(\tilde{\Sigma}, \tilde{P})$ be the Hitchin moduli space over a Riemann surface thus is hyperkahler on the smooth part with three symplectic forms Ω_1, Ω_2, Ω_3 defined earlier. Consider an involution τ on the moduli space $\mathcal{M}(\tilde{\Sigma}, \tilde{P})$ (notice that there are many involutions on the moduli space induced by the deck transformation) induced by the deck transformation on the Riemann surface $\tilde{\Sigma}$ (which is orientation reversing and without fixed points). Then the involution τ is a symplectic isometry with respect to Ω_3 but anti-symplectic with respect to Ω_1 and Ω_2.*

Proof. Using the fact that τ is an orientation reversing map and $\tau^*\star = -\star\tau^*$, a direct calculation shows:

$$\begin{aligned}
\tau^*\omega_1((\theta, \xi), (\vartheta, \eta)) = \omega_1((\tau^*\theta, \tau^*\xi), (\tau^*\vartheta, \tau^*\eta)) &= \int_{\tilde{\Sigma}} \mathrm{Tr}\ \tau^*\theta \wedge \tau^*\vartheta - \tau^*\xi \wedge \tau^*\eta \\
&= \int_{\tilde{\Sigma}} \mathrm{Tr}\ \tau^*(\theta \wedge \vartheta - \xi \wedge \eta) = \int_{\tilde{\Sigma}} -\mathrm{Tr}\ (\theta \wedge \vartheta - \xi \wedge \eta) \\
&= -\omega_1((\theta, \xi), (\vartheta, \eta));
\end{aligned}$$

$$\begin{aligned}\tau^*\omega_2((\theta,\xi),(\vartheta,\eta)) &= \omega_2((\tau^*\theta,\tau^*\xi),(\tau^*\vartheta,\tau^*\eta)) = \int_{\tilde\Sigma} \mathrm{Tr}\ \tau^*\theta\wedge\tau^*\eta - \tau^*\vartheta\wedge\tau^*\xi \\ &= \int_{\tilde\Sigma} \mathrm{Tr}\ \tau^*(\theta\wedge\eta-\vartheta\wedge\xi) = \int_{\tilde\Sigma} -\mathrm{Tr}\ (\theta\wedge\eta-\vartheta\wedge\xi) \\ &= -\omega_2((\theta,\xi),(\vartheta,\eta));\end{aligned}$$

$$\begin{aligned}\tau^*\omega_3((\theta,\xi),(\vartheta,\eta)) &= \omega_3((\tau^*\theta,\tau^*\xi),(\tau^*\vartheta,\tau^*\eta)) = \int_{\tilde\Sigma} \mathrm{Tr}\ \tau^*\theta\wedge\star\tau^*\eta - \tau^*\xi\wedge\star\tau^*\vartheta \\ &= \int_{\tilde\Sigma} \mathrm{Tr}\ \tau^*\theta\wedge(-\tau^*\star\eta) - \tau^*\xi\wedge(-\tau^*\star\vartheta) \\ &= \int_{\tilde\Sigma} -\mathrm{Tr}\ \tau^*(\theta\wedge\star\eta-\xi\wedge\star\vartheta) = \int_{\tilde\Sigma} \mathrm{Tr}\ \theta\wedge\eta-\xi\wedge\star\vartheta \\ &= \omega_3((\theta,\xi),(\vartheta,\eta)).\end{aligned}$$

Thus τ is anti-symplectic with respect to ω_1, ω_2 and symplectic with respect to ω_3. Now since Ω_1, Ω_2 and Ω_3 were induced by ω_1, ω_2 and ω_3 via the symplectic reduction, τ is anti-symplectic with respect to Ω_1, Ω_2 and symplectic with respect to Ω_3. $\square$

Recall in Section 4, Theorem 4.9 shows that the natural map

$$I : \bigcup_{[P]} \mathcal{M}(\Sigma,P)^{\mathrm{smooth}} \to (\bigcup_{[\tilde P]} \mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}})^\tau$$

is a covering map onto $\mathcal{N}_0 \subset (\mathrm{Hom}(\pi_1(\tilde\Sigma),G)^{\mathrm{simp}}/\!\!/G)^\tau \cong (\bigcup_{[\tilde P]} \mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}})^\tau$. Thus, we have:

Theorem 5.4. *The Hitchin moduli space $\mathcal{M}(\Sigma,P)$ has a natural symplectic structure on the smooth part.*

Proof. Following Proposition 5.2 and Lemma 5.3, we see that the fixed point set $(\mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}})^\tau \subset (\mathrm{Hom}(\pi_1(\tilde\Sigma),G)^{\mathrm{simp}}/\!\!/G)^\tau$ is symplectic. The image under I of the smooth moduli space $I(\mathcal{M}(\Sigma,P)^{\mathrm{smooth}}) = \mathcal{N}_0 \cap \mathcal{M}(\tilde\Sigma,\tilde P)^\tau$ is a union of components of $(\mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}})^\tau$. Thus we obtain a symplectic structure on $\mathcal{M}(\Sigma,P)^{\mathrm{smooth}}$ under the pull back of the finite covering map I. $\square$

Remark 5.5. *A priori, we only know that the dimension of the moduli space $\mathcal{M}(\Sigma,P)^{\mathrm{smooth}}$ is even by Proposition 5.2. However, since Lemma 5.3 implies that $(\mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}})^\tau$ is a Lagrangian submanifold of $\mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}}$ with respect to both Ω_1 and Ω_2, so the dimension of $\mathcal{M}(\Sigma,P)^{\mathrm{smooth}}$ must be half of the dimension of $\mathcal{M}(\tilde\Sigma,\tilde P)^{\mathrm{smooth}}$.*

5.2. **Symplectic reduction.** Another view point is to realize the Hitchin moduli spaces over nonorientable surfaces as a symplectic quotient.

Let us consider a Hamiltonian G-space (M,ω,μ) where G is a compact Lie group and the action is free. Let $\tau : M \to M$ and $\sigma : G \to G$ be two involutions. The

involution $\sigma : G \to G$ induces an involution on $\mathfrak{g}$ and an involution on $\mathfrak{g}^*$ both of which, for convenience, we will denote by σ also. Clearly, both G^σ and M^τ could be empty and even when they are nonempty, they may be disconnected.

Lemma 5.6. *Let $\mathfrak{g}^\pm \subset$ be the ± 1-eigenspace of $\sigma : \mathfrak{g} \to \mathfrak{g}$ and $(\mathfrak{g}^*)^\pm$ the ± 1-eigenspace of $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*$. Then $(\mathfrak{g}^\pm)^* \cong (\mathfrak{g}^*)^\pm$.*

Proof. Since $\mathfrak{g}^\pm \subseteq \mathfrak{g}$ are the ± 1-eigenspace of $\sigma : \mathfrak{g} \to \mathfrak{g}$, then $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ and $\mathfrak{g}^+ \cong \mathrm{Lie}((G^\sigma)_o) \cong \mathrm{Lie}(G^\sigma)$, where $(G^\sigma)_o$ is the identity component of G^σ. Now if we identify the annihilator of $\mathfrak{g}^\pm$ with the dual space $(\mathfrak{g}^\mp)^*$ then we have $\mathfrak{g}^* = (\mathfrak{g}^-)^* \oplus (\mathfrak{g}^+)^*$. Clearly $\mathfrak{g}^*$ has an eigenspace-decomposition $\mathfrak{g}^* = (\mathfrak{g}^*)^+ \oplus (\mathfrak{g}^*)^-$ with respect to the involution $\sigma : \mathfrak{g}^* \to \mathfrak{g}^*$. Let $\xi = \xi^+ \oplus \xi^- \in \mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. If $f \in (\mathfrak{g}^-)^*$, then $\sigma(f)(\xi) = f(\sigma(\xi^+) + \sigma(\xi^-)) = f(\xi^+) - f(\xi^-) = -f(\xi^+) - f(\xi^-) = -f(\xi)$ where we used $f(\xi^+) = 0 = -f(\xi^+)$ since the annihilator of $\mathfrak{g}^+$ is identified with the dual space $(\mathfrak{g}^-)^*$. Thus $f \in (\mathfrak{g}^*)^-$. Similarly, if $f \in (\mathfrak{g}^+)^*$ then $f \in (\mathfrak{g}^*)^+$, i.e. $(\mathfrak{g}^\pm)^* \subseteq (\mathfrak{g}^*)^\pm$. On the other hand, if $f \in (\mathfrak{g}^*)^+$ and $\xi \in \mathfrak{g}^-$ then $f(\xi^-) = \sigma(f)(\xi^-) = f(\sigma(\xi^-)) = f(-\xi^-) = -f(\xi^-)$, i.e. $f(\xi^-) = 0$ for all $\xi^- \in \mathfrak{g}^-$. In other words, f is in the annihilator of $\mathfrak{g}^-$ i.e. $f \in (\mathfrak{g}^+)^*$. Similarly, if $f \in (\mathfrak{g}^*)^-$ then $f \in (\mathfrak{g}^-)^*$. Thus $(\mathfrak{g}^*)^\pm \subseteq (\mathfrak{g}^\pm)^*$ which now implies that $(\mathfrak{g}^*)^\pm \cong (\mathfrak{g}^\pm)^*$.

Notice that if G is finite dimensional, the proof can be simplified: since $(\mathfrak{g}^*)^+ \oplus (\mathfrak{g}^*)^-$ and $(\mathfrak{g}^+)^* \oplus (\mathfrak{g}^-)^*$ are two decompositions of the same space $\mathfrak{g}^*$, together with $(\mathfrak{g}^\pm)^* \subseteq (\mathfrak{g}^*)^\pm$ we conclude that $(\mathfrak{g}^\pm)^* \cong (\mathfrak{g}^*)^\pm$ for dimension reason. ☐

We say that these two involutions τ and σ are *compatible with the G-action* if the following two statements are true: (cf. [OS])
(1) $\tau(g \cdot m) = \sigma(g) \cdot \tau(m)$; and
(2) $\mu(\tau(m)) = \sigma(\mu(m))$, when $\tau : M \to M$ is symplectic
or
(2') $\mu(\tau(m)) = -\sigma(\mu(m))$, when $\tau : M \to M$ is anti-symplectic.

Theorem 5.7. *Given a Hamiltonian G-space (M, ω, μ) where G is compact and the action is free. Let $\tau : M \to M$ be a symplectic involution and $\sigma : G \to G$ be an involution. Assume that they are compatible with the G-action and that M^τ and G^σ are nonempty. Then $(M^\tau, \imath^*\omega, \mu|_{M^\tau})$ is a Hamiltonian G^σ-space, where $\imath : M^\tau \to M$ is the inclusion. In particular, 0 is a regular value of $\mu|_{M^\tau}$, and $(\mu|_{M^\tau})^{-1}(0)/G^\sigma$ is a symplectic manifold.*

Proof. To show that G^σ is a compact Lie group, we only need to prove that G^σ is a closed subgroup of G. Let $x,\ y \in G^\sigma$,

$$\sigma(xy)\tau(m) = \tau(xym) = \sigma(x)\tau(ym) = \sigma(x)\sigma(y)\tau(m) = xy\tau(m)$$

for all $m \in M$. Thus $(xy)^{-1}\sigma(xy) \in \cap_{m\in M} G_{\tau(m)}$. Since the action is free, we see that $\sigma(xy) = xy$ and $xy \in G^\sigma$. In other words, G^σ is a subgroup of G. Given a sequence $\{x_n\} \in G^\sigma$. Assume that $x \in G$ is the limit point of the sequence. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sigma(x_n) = \sigma(\lim_{n\to\infty} x_n) = \sigma(x)$$

since σ is continuous. Thus, G^σ is closed. Moreover, since G action on M is free, G^σ action on M^τ is also free.

If $m \in M^\tau$ then $\sigma(\mu(m)) = \mu(\tau(m)) = \mu(m)$, i.e. $\mu(m) \in (\mathfrak{g}^*)^+ \cong (\mathfrak{g}^+)^* \cong (\mathrm{Lie}(G^\sigma_o))^* \cong (\mathrm{Lie}(G^\sigma))^*$. Thus $\mu(M^\tau) \subseteq (\mathfrak{g}^+)^*$ and $\mu\mid_{M^\tau}$ is G^σ-equivariant since μ is G-equivariant.

If $v \in T_m M^\tau$ then $v = r'(0)$ for some $r(t) \in M^\tau$. Let $g \in G^\sigma$, then $\tau(gr(t)) = \sigma(g)\tau(r(t)) = \sigma(g)r(t) = gr(t)$, meaning $gr(t) \in M^\tau$ and $g_*(v) \in T_m M^\tau$. Thus, $(\imath^*\omega)_m(v_1, v_2) = \omega_m(\imath_* v_1, \imath_* v_2) = \omega_{g.m}(g_*\imath_* v_1, g_*\imath_* v_2) = \omega_{g.m}(\imath_* g_* v_1, \imath_* g_* v_2) = (\imath^*\omega)_{g.m}(g_* v_1, g_* v_2)$ for all $g \in G^\sigma$ where we used that ω is G-invariant, i.e. $g^*\omega = \omega$ for all $g \in G$. In other words, $\imath^*\omega$ is G^σ-invariant.

For all $\xi \in \mathfrak{g}^+$ and $X \in T_m M^\tau$,

$$(\imath^*\omega)_m(\xi^{M^\tau}(m), X) = \omega_m(\xi^M(m), X) = < d\mu_m(X), \xi >,$$

since $\imath$ is an inclusion and the original action is Hamiltonian. Thus, we have shown that $(M^\tau, \imath^*\omega, \mu|_{M^\tau})$ is a Hamiltonian G^σ-space.

Since G^σ-action is free and G^σ is compact, we have that 0 is also a regular value of $\mu|_{M^\tau}$ and $(\mu|_{M^\tau})^{-1}(0)/G^\sigma$ is a symplectic manifold. □

Remark 5.8. *The compatibility of the two involutions τ and σ with the G-action only implies that $\tau(\mu^{-1}(0)) \subset \mu^{-1}(0)$, but it does not imply that $\mu^{-1}(0) \cap M^\tau \neq \emptyset$. Nevertheless, in the case of the Hitchin moduli space studied in this paper, this intersection is always nonempty.*

Remark 5.9. *Notice that, even if the G-action is not free, the assumption that 0 is a regular value of μ can still imply that 0 is a regular value of $\mu|_{M^\tau}$.*

Example 5.10. *Consider $(\mathbb{C}^3, \omega, \Phi)$ a Hamiltonian $(S^1)^{\times 3}$-space where*

$$(t_1, t_2, t_3) \cdot (z_1, z_2, z_3) = (t_1 z_1, t_2 z_2, t_3 z_3)$$

$$\Phi(z_1, z_2, z_3) = \frac{1}{2}(|z_1|^2, |z_2|^2, |z_3|^2)$$

$$\omega = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$$

Consider an involution $\tau : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $(z_1, z_2, z_3) \mapsto (z_1, z_3, z_2)$ and an involution $\tau : (S^1)^{\times 3} \to (S^1)^{\times 3}$ by $(t_1, t_2, t_3) \mapsto (t_1, t_3, t_2)$.

Clearly, τ is symplectic $\tau^*\omega = \omega$ and the group action is compatible with both involutions $\tau\left((t_1, t_2, t_3) \cdot (z_1, z_2, z_3)\right) = \tau\left((t_1, t_2, t_3)\right) \cdot \tau\left((z_1, z_2, z_3)\right)$.

The fixed point sets $M^\tau = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_2 = z_3\} \cong \mathbb{C}^2$, *and* $G^\tau = \{(t_1, t_2, t_3) \in (S^1)^3 \mid t_2 = t_3\} \cong (S^1)^{\times 2}$. *The pull back symplectic form on* $\mathbb{C}^2$ *is now*

$$\frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 + \sqrt{-1} dz_2 \wedge d\bar{z}_2,$$

and the moment map with respect to the $(S^1)^{\times 2}$ *action is now*

$$\Phi(z_1, z_2) = (\frac{1}{2}|z_1|^2, |z_2|^2).$$

Recall that a flat connection on a Riemann surface is called simple if it has minimal isotropy. We define a flat connection on a nonorientable surface to be simple if its lifting to the orientable double cover is simple. Notice that this definition is compatible with the definition for simple representations in the Betti moduli space defined in Section 4. In the companion paper [HW], we will give a different definition for simple connections on a nonorientable surface and show that the definition here and the definition there are equivalent and that they correspond precisely to the smooth points on the de Rham moduli space, when they are also reductive.

We then define the following subspaces of $T^*\mathcal{A}(\tilde{P})$ over $\tilde{\Sigma}$:

$$F_{\tilde{\Sigma}}(\tilde{P}) = \{(\theta, \xi) \in T^*\mathcal{A}(\tilde{P}) |\ Curv(\theta + \sqrt{-1}\xi) = 0\},$$
$$F_{\tilde{\Sigma}}^{\text{simp}}(\tilde{P}) = \{(\theta, \xi) \in F_{\tilde{\Sigma}}(\tilde{P}) |\ \theta + \sqrt{-1}\xi \text{ is simple}\};$$
$$F_{\tilde{\Sigma}}^{\text{red}}(\tilde{P}) = \{(\theta, \xi) \in F_{\tilde{\Sigma}}(\tilde{P}) |\ \theta + \sqrt{-1}\xi \text{ is reductive}\};$$
$$F_{\tilde{\Sigma}}^{\text{smooth}}(\tilde{P}) = \{(\theta, \xi) \in F_{\tilde{\Sigma}}^{\text{red}}(\tilde{P}) \cap F_{\tilde{\Sigma}}^{\text{simp}}(\tilde{P})\}$$

and similarly define some subspaces of $T^*\mathcal{A}(P)$ over Σ:

$$F_{\Sigma}(P) = \{(A, \alpha) \in T^*\mathcal{A}(P) |\ Curv(A + \sqrt{-1}\alpha) = 0\}$$
$$F_{\Sigma}^{\text{simp}}(P) = \{(A, \alpha) \in F_{\Sigma}(P) | A + \sqrt{-1}\alpha \text{ is simple}\}$$
$$F_{\Sigma}^{\text{red}}(P) = \{(A, \alpha) \in F_{\Sigma}(P) | A + \sqrt{-1}\alpha \text{ is reductive}\}$$
$$F_{\Sigma}^{\text{smooth}}(P) = \{(A, \alpha) \in F_{\Sigma}^{\text{red}}(P) \cap F_{\Sigma}^{\text{simp}}(P)\}$$

where $\theta + \sqrt{-1}\xi$ and $A + \sqrt{-1}\alpha$ are viewed as $K^{\mathbb{C}}$ connections. Notice that we use the terminology *smooth* and not *irreducible* here because while irreducible connections on a Riemann surface correspond to the smooth points in the moduli space, irreducible connections on a nonorientable surface are not necessarily the smooth points in the moduli space, and they don't always lift to irreducible connections on a Riemann surface either. This will be explained in detail in [HW].

It was explained in Section 2 that the deck transformation $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ induces involutions $\tau : \tilde{P} \to \tilde{P}$ such that $P \cong \tilde{P}/\tau$ and thus isomorphisms between the following spaces:

$$\mathcal{G} \cong \tilde{\mathcal{G}}^\tau, \quad F_\Sigma \cong F_{\tilde{\Sigma}}^\tau, \quad F_\Sigma^{\text{simp}} \cong (F_{\tilde{\Sigma}}^{\text{simp}})^\tau,$$
$$F_\Sigma^{\text{red}} \cong (F_{\tilde{\Sigma}}^{\text{red}})^\tau, \quad F_\Sigma^{\text{smooth}} \cong (F_{\tilde{\Sigma}}^{\text{smooth}})^\tau.$$

The following theorem is then a corollary of Theorem 5.7.

Theorem 5.11. *The smooth Hitchin moduli space $\mathcal{M}(\Sigma, P)^{\text{smooth}}$ over a nonorientable surface Σ is a symplectic manifold and can be viewed as a symplectic quotient.*

Proof. Recall that the smooth Hitchin moduli space $\mathcal{M}(\tilde{\Sigma}, \tilde{P})^{\text{smooth}}$ over a Riemann surface $\tilde{\Sigma}$ can be constructed by taking the symplectic reduction of $F_{\tilde{\Sigma}}^{\text{simp}}(\tilde{P})$ with symplectic form ω_3 by $\tilde{\mathcal{G}}$ at the zero level set of the moment map μ_3 (cf. [Hi1, Fu]). The involution τ induced by the deck transformation acts on $F_{\tilde{\Sigma}}^{\text{simp}}$ and $\tilde{\mathcal{G}}$ and is compatible with the $\tilde{\mathcal{G}}$ action. Now since the center of the gauge group acts trivially on connections and the isotropy at simple points of the gauge group is precisely the center, so the action of the gauge group modulo the center is free and the moduli space by this quotient group is the same as the moduli space by the whole gauge group. Thus, the gauge action can be viewed as free action on the set of flat simple points. Thus, Theorem 5.7 tells us that the action of $\tilde{\mathcal{G}}^\tau$ on $((F_{\tilde{\Sigma}}^{\text{simp}})^\tau, \imath^*\omega_3)$ is Hamiltonian with moment map $\mu_3|_{(F_{\tilde{\Sigma}}^{\text{simp}})^\tau} \stackrel{\text{def}}{=} \mu(A, \alpha) = d_A^*\alpha$. Recall that $d_A^*\alpha$ on $T^*\mathcal{A}(P) \cong (T^*\mathcal{A}(\tilde{P}))^\tau$ was defined in definition 2.3. The zero level set in $(F_{\tilde{\Sigma}}^{\text{simp}})^\tau$ of the moment map $\mu_3|_{(F_{\tilde{\Sigma}}^{\text{simp}})^\tau}$ is precisely the set of solutions to the Hitchin's equations on the nonorientable surface Σ (cf: Section 3) that are simple. Thus, by taking the quotient of $(\mu_3|_{(F_{\tilde{\Sigma}}^{\text{simp}})^\tau})^{-1}(0)$ by $\tilde{\mathcal{G}}^\tau$, we arrive at the smooth Hitchin moduli space $\mathcal{M}(\Sigma, P)^{\text{smooth}}$ over a nonorientable surface Σ which, by Theorem 5.7, is constructed as a Marsden-Weinstein quotient and thus is symplectic. $\square$

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